Lesson 6: Mathematical Models of Fluid Flow Components

ET 438a

Automatic Control Systems Technology

Learning Objectives

After this presentation you will be able to:

- Define the characteristics of a fluid flow system
- Identify if a fluid flow is Laminar or Turbulent based on fluid and system parameters.
- Write mathematical models for fluid characteristics.
- Develop an analogy between electrical characteristic and fluid system characteristics.
- Solve for steady-state fluid flow using given mathematical modeling equations.

Liquid Flow Characteristics

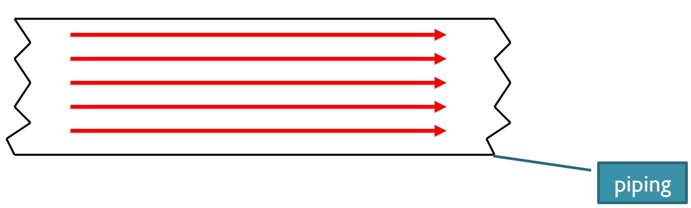
Flow	Volume of liquid/unit time (e.g. gallons/minute)	Current (Coulombs/s)
Pressure	force/unit area (e.g. lb/in^2)	Potential Difference
Flow Resistance	Pressure drop/unit flow (e.g. $lb/in^2/gal$)	Electrical Resistance

Note: Flow resistance is either linear or non-linear. It depends on type of fluid flow, which is based on the fluid and piping parameters.

Fluid Flow Classifications

Types of liquid flow

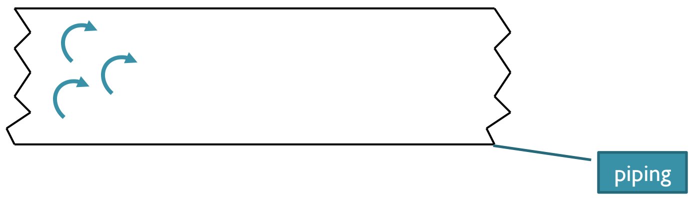
Laminar Flow - low velocity flows. Stream lines are parallel. Liquid flows in layers. Linear flow resistance.



Fluid Flow Classifications

Types of liquid flow

Turbulent - relatively high velocity flow. Liquid swirls and spins as it flows. Non-linear flow resistance.



Flow type determined by the Reynold's Number

Determining Flow Types

Reynold's Number

$$R = \frac{\rho \cdot v \cdot d}{\mu}$$

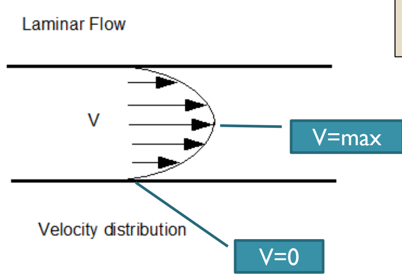
Where ρ = density of the fluid (kg/m^3)
v = average velocity of the fluid (m/s)
d = diameter of pipe (m)
μ = absolute viscosity of fluid (Pa-s)

Note: Reynold's number is dimensionless

Laminar flow : R<2000
Turbulent flow: R>4000
Transition flow: 2000< R< 4000

Determining Flow Types

Computing average velocity



Velocity profile changes across the cross section of pipes and ducts

$$v = \frac{Q}{A}$$

Since pipe diameter is usually given

$$v = \frac{4 \cdot Q}{\pi \cdot d^2}$$

Where: A = area of pipe (m^2)
Q = flow (m^3/s)
d = pipe diameter (m)

Laminar Flow Equations

Laminar Flow Equations for Round Pipes

$$P = R_L \cdot Q \quad \text{(Pa)}$$

$$R_L = \frac{128 \cdot \mu \cdot l}{\pi \cdot d^4}$$

Where: p = pressure drop (Pascals)
R_L = laminar flow resistance
Q = flow (m^3/s)
l = length of pipe (m)
m = absolute viscosity (Pa-s)
d = pipe diameter (m)

Turbulent Flow Equations

Turbulent Flow Equations for Round Pipes

$$P = K_t \cdot Q^2 \qquad (\text{Pa})$$
$$K_t = \frac{8 \cdot \rho \cdot f \cdot l}{\pi^2 \cdot d^5} \qquad (\text{Pa-s/m}^3)$$
$$R_t = 2 \cdot K_t \cdot Q$$

Where: f= friction factor (see table 3.3 p. 81 text)
l = length (m)
d = pipe diameter (m)
ρ = density of liquid (kg/m^3)
R_t = turbulent flow resistance (Pa-s/m^3)
p = pressure (Pa)
Q = flow (m^3/s)

Laminar Flow Example

Example 6-1: Oil at a temperature of 15 C flows in a horizontal, 1 cm diameter tube with a flow rate of 9.42 L/min. Tube length is 10 m. Find: Reynold's number, flow resistance, pressure drop in tube.

Convert all units into SI units

$d = 1\,cm = 0.01\,m$

$Q = 9.42\,L/min\,(1.6667\times10^{-5}\,m^3/s/L/min)$

$Q = 1.57\times10^{-4}\,m^3/s$

All conversion factors are in the textbook appendix

Average velocity $V = \frac{4Q}{\pi d^2} = \frac{4(1.57\times10^{-4}\,m^3/s)}{\pi(0.01\,m)^2} = 2.0\,m/s$

From Appendix A in textbook ρ=880 kg/m^3 μ=0.160 Pa-s

Compute Reynold's number $R = \frac{\rho V d}{\mu} = \frac{(880\,kg/m^3)(2\,m/s)(0.01\,m)}{(0.160\,Pa\text{-}s)} = \boxed{109.95}$

R<2000 so flow in Laminar. Linear relationship between flow and pressure drop.

Laminar Flow Solution (2)

Compute Laminar flow resistance

$$R_L = \frac{128\mu l}{\pi d^4}$$

$$R_L = \frac{128(0.160\ Pa\text{-}s)(10m)}{\pi(0.01)^4}$$

$$\boxed{R_L = 6.519\times10^9\ Pa\text{-}s/m^3}$$

Compute pressure drop

$$p = R_L Q$$

$$p = (6.519\times10^9\ Pa\text{-}s/m^3)(1.57\times10^{-4}\ m^3/s)$$

$$\boxed{p = 1.0285\times10^6\ Pa}$$

Convert to psi $\quad p = (1.0285\times10^6\ Pa)(1.45\times10^{-4}\ psi/Pa) = \boxed{148.4\ psi}$

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Turbulent Flow Example

Example 6-2: Water at 15 C flows through a commercial steel pipe with a diameter of 0.4 inch with a flow rate of 6 gal/min. The line is 50 ft long. Find: Reynold's number, flow resistance, pressure drop in pipe.

From Appendix A Density ρ=1000 kg/m^3 Viscosity μ=0.001 Pa-s

Convert all English units to SI units

$$d = (0.4\ in)(0.0254\ m/in) = 0.01016\ m$$

$$Q = (6\ gal/min)(6.3088\times10^{-5}\ m^3/s/gpm) = 3.7853\times10^{-4}\ m^3/s$$

$$l = (50\ ft)(0.3048\ m/ft) = 15.24\ m$$

Find the average velocity

$$V = \frac{4Q}{\pi d^2} = \frac{4(3.7853\times10^{-4}\ m^3/s)}{\pi(0.0106m)^2} \qquad V = 4.669\ m/s$$

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Turbulent Flow Solution (2)

Compute the Reynold's number

$$R = \frac{\rho V d}{\mu} = \frac{(1000\ Kg/m^3)(4.669\ m/s)(0.0106\ m)}{0.001\ Pa\text{-}s}$$

$$R = 47{,}440 \quad R > 4000$$

R> 2000 so the flow is turbulent. Use the turbulent equations.

Turbulent equations require friction factors from Table 3-3 on page 81 of textbook.

$$P = K_t Q^2$$

$$R_t = 2K_t Q$$

$$K_t = \frac{8 \rho f L}{\pi^2 d^5}$$

Find f values from Table 3-3. Get values that bracket the Computed value.

For commercial steel pipe: diameter 1-2 cm

R between	$R_a > 10{,}000$	$R_b < 100{,}000$
f between	$f_a = 0.035$	$f_b = 0.028$

Turbulent Flow Solution (3)

Friction factor, f, must be between f_a and f_b

$$f = 0.035 + (0.028 - 0.035)\left[\frac{47{,}440 - 10{,}000}{100{,}000 - 10{,}000}\right]$$

$$f = 0.035 + -0.002\left(\frac{3.744}{9}\right)$$

$$f = 0.03209$$

Now find K.

$$K_t = \frac{8 \rho f L}{\pi^2 d^5} = \frac{8(1000\ Kg/m^3)(0.03209)(15.24\ m)}{\pi^2 (0.01016\ m)^5}$$

$$K_t = 3.6614 \times 10^{12}\ Pa\text{-}s/m^3$$

Find R_t

$$R_t = 2K_t Q$$

$$R_t = 2(3.6614 \times 10^{12}\ Pa\text{-}s/m^3)(3.7853 \times 10^{-4}\ m^3/s)$$

$$\boxed{R_t = 2.7719 \times 10^{9}\ Pa\text{-}s/m^3}$$

Turbulent Flow Solution (4)

Convert R_t to English units

$$R_t = (2.7719 \times 10^9\ Pa\text{-}s/m^3)(9.148 \times 10^{-9}) = \boxed{25.357\ psi/gpm}$$

From Appendix

Compute pressure drop

$$P = K_t Q^2$$
$$P = (3.6614 \times 10^{12}\ Pa\text{-}s/m^6)(3.7853 \times 10^{-4}\ m^3/s)^2$$
$$P = 5.246 \times 10^5\ Pa$$

Convert P to English units

$$P = (5.246 \times 10^5\ Pa)(1.45 \times 10^{-4}\ psi/Pa)$$
$$\boxed{P = 76.07\ psi}$$

Liquid Flow Capacitance

Liquid Flow Capacitance - increase in volume of liquid required to produce unit increase in pressure

$$C_L = \frac{\Delta V}{\Delta P}$$

Where: C_L = capacitance (m^3/Pa)
ΔV = volume change (m^3)
Δp = pressure change (Pa)

Derive relationship for C_L.

Pressure relationship $\Delta P = \rho \cdot g \cdot \Delta H$

Where: ρ = density of fluid
g = acceleration due to gravity
H = height of liquid in tank

Liquid Flow Capacitance Derivation

Find liquid flow capacitance in terms of tank parameters.

$$\Delta P = \rho \cdot g \cdot \Delta H$$

$$\Delta H = \frac{\Delta V}{A}$$

$$\Delta P = \rho \cdot g \cdot \left(\frac{\Delta V}{A}\right)$$

$$C_L = \frac{\Delta V}{\Delta P} = \frac{\Delta V}{\rho \cdot g \cdot \left(\frac{\Delta V}{A}\right)} = \frac{A}{\rho \cdot g} \ (m^3/Pa)$$

$$C_L = \frac{A}{\rho \cdot g} \ (m^3/Pa)$$

Final Equation

Fluid Capacitance Examples

Example 6-3: A tank has a diameter of 1.83 meters and a height of 10 ft. Determine the capacitance of the tank when it holds: a.) water b.) oil c.) kerosene d.) gasoline

a.) $C_L = \frac{A}{\rho g}$ $A = \frac{\pi d^2}{4} = \frac{\pi (1.83\,m)^2}{4}$

$A = 2.63\,m^2$

$\rho = 1000\,kg/m^3$ $C_L = \frac{2.63\,m^2}{1000\,kg/m^3\ 9.81\,m/s^2}$

$\boxed{C_L = 2.68 \times 10^{-4}\,m^3/Pa}$

b.) $\rho = 880\,kg/m^3$ $C_L = \frac{2.63\,m^2}{(880\,kg/m^3)(9.81\,m/s^2)}$

$\boxed{C_L = 3.05 \times 10^{-3}\,m^3/Pa}$

Fluid Capacitance Examples

Parts c. and d.

c.) $\rho = 800\ Kg/m^3$

$$C_L = \frac{2.63\ m^2}{(800\ Kg/m^3)(9.81\ m/s^2)}$$

$$\boxed{C_L = 3.35 \times 10^{-4}\ m^3/Pa}$$

d.) $\rho = 740\ Kg/m^3$

$$\frac{2.63\ m^2}{(740\ Kg/m^3)(9.801\ m/s^2)} = C_L$$

$$\boxed{3.62 \times 10^{-4}\ m^3/Pa = C_L}$$

Note: As density of fluid decreases, the volume of liquid required to produce a unit increase in pressure increases.

Fluid Inertance

Amount of pressure drop required to increase flow rate by one unit/second. Analogy-electrical inductance.

$$I_L = \frac{p}{\frac{\Delta Q}{\Delta t}}$$

Where I_L = inertance ($Pa/(m^3/s^2)$)
p = pressure drop (Pa)
$\Delta Q/\Delta t$ = change in flow

Inertance defined using physical parameters of pipe.

$$I_L = \frac{\rho \cdot l}{A}$$

Where: A = area of pipe (m^2)
ρ = density of liquid (Kg/m^3)
l = length of pipe (m)

Dead-Time Delay

Dead-time Delay of Liquid - time required to transport liquid from one point to another in piping system or ducts.

$$t_d = \frac{D}{v}$$

v = average velocity of fluid (m/s)
D = distance traveled (m)

Example 6-4: Determine the inertance of water in a pipe with a diameter of 2.1 cm and a length of 65 meters.

$d = 2.1\text{cm} = 0.021\text{m}$ $\quad I_L = \frac{\rho l}{A}$ $\quad A = \frac{\pi(0.021)^2}{4} = 3.46\times10^{-4}\text{m}^2$

$\rho = 1000\text{ kg/m}^3$ $\quad I_L = \frac{(1000\text{ kg/m}^3)(65\text{m})}{3.46\times10^{-4}\text{m}^2}$

$l = 65\text{m}$ $\quad A = \frac{\pi d^2}{4}$

$$\boxed{I_L = 1.88\times10^8\ \text{Pa}/\text{m}^3/\text{s}^2}$$

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END LESSON 6: MATHEMATICAL MODELS OF FLUID FLOW COMPONENTS