Lesson 12: Transfer Functions In The Laplace Domain

ET 438a Automatic Control Systems Technology

Learning Objectives

After this presentation you will be able to:

- Define the terms pole, zero and eigenvalue as they pertain to transfer functions.
- Identify the location of poles and zeros on the complex plane.
- Develop transfer functions from OP AMP circuits using the Laplace variable.
- Develop transfer functions of electromechanical systems using the Laplace variable.
- Find the values of poles and zeros given a transfer function.

Definition of Transfer Function

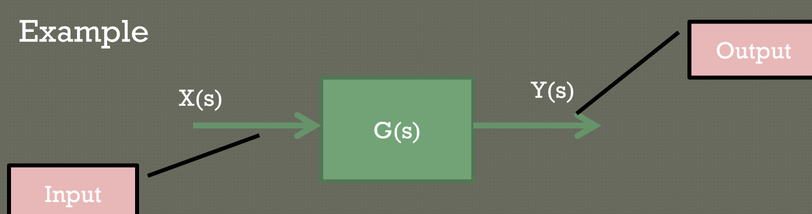
Input/output relationships for a mathematical model usually given by the ratio of two polynomials of the variable s.

Where $$G(s) = \frac{a_n \cdot s^n + a_{n-1} + a_{n-2} \cdot s^{n-3} \ldots + a_2 \cdot s^2 + a_1 \cdot s + a_0}{b_n \cdot s^n + b_{n-1} + b_{n-2} \cdot s^{n-3} \ldots + b_2 \cdot s^2 + b_1 \cdot s + b_0}$$

All a's and b's are constants
Order of numerator is less that the denominator

Definition of Transfer Function

Example



$$Y(s)= G(s)\cdot X(s)$$

Transfer function is the "gain" of the block as a function of the Laplace variable s.

Transfer Function Terminology

Definitions:

Poles - roots of the denominator polynomial. Values that caus transfer function magnitude to go to infinity.

Zeros - roots of the numerator polynomial. Values that cause the transfer function to go to 0.

Eigenvalues - Characteristic responses of a system. Roots of the denominator polynomial. All eigenvalues must be negative for a system transient (natural response) to decay out.

Zeros

Poles

$$G(s) = \frac{(s+z_1)\cdot(s+z_2)\cdot(s+z_3)\ldots\ \cdot(s+z_{n-1})\cdot(s+z_n)}{(s+p_1)\cdot(s+p_2)\cdot(s+p_3)\ldots\ \cdot(s+p_{n-1})\cdot(s+p_n)}$$

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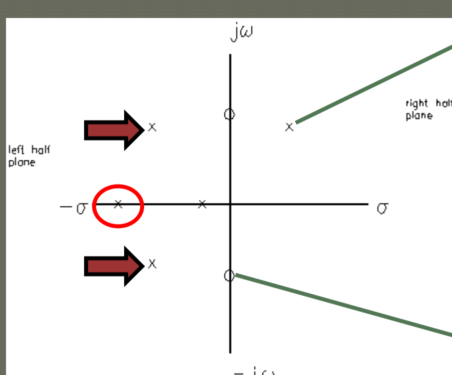
Pole/Zero Plots

Transfer function poles and zeros determine systems' responses. Plotted on the complex plane (s,jω).



X's indicate pole location

Closer pole is to imaginary axis slower response.

Complex roots appear in conjugate pairs

Circle is location of zero

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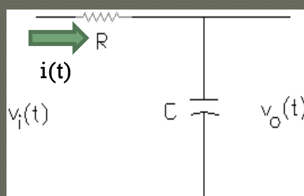
Transfer Function Examples

Example 12-1: Find the transfer function of the low pass filter shown below. Draw a block diagram of the result showing the input/output relationship.



Write a KVL equation around the RC loop.

$$v_i(t) = R \cdot i(t) + \frac{1}{C} \cdot \int i(t)\, dt$$

Take Laplace Transform of the above equation

$$V_i(s) = R \cdot I(s) + \frac{1}{Cs} \cdot I(s)$$

Now find the current from the above equation

Transfer Function Example 12-1 (1)

Solve for I(s)

$$V_i(s) = R \cdot I(s) + \frac{1}{Cs} \cdot I(s)$$

$$V_i(s) = (R + \frac{1}{Cs}) \cdot I(s)$$

Factor out I(s)

$$\frac{V_i(s)}{(R + \frac{1}{Cs})} = I(s)$$

Divide both sides by (R+1/Cs)

Remember

$$V_o(s) = \frac{1}{Cs} \cdot I(s)$$

Substitute I(s) into above equation and simplify

$$V_o(s) = \left[\frac{\frac{1}{Cs} \cdot V_i(s)}{(R + \frac{1}{Cs})} \right]$$

$$V_o(s) = \frac{Cs}{Cs} \cdot \left[\frac{\frac{1}{Cs} \cdot V_i(s)}{(R + \frac{1}{Cs})} \right]$$

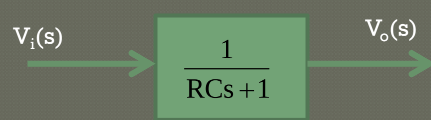
$$(RCs + 1)$$

Transfer Function Example 12-1 (2)

Final formula

$$V_o(s) = \left[\frac{1}{(RCs+1)}\right] \cdot V_i(s)$$

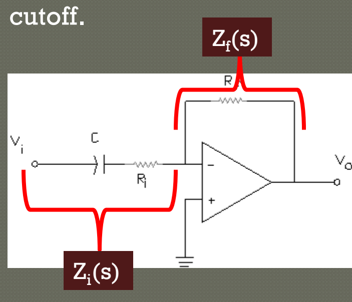
Draw block diagram



RC is time constant of system.
System has 1 pole at -1/RC and no zeros.
Larger RC gives slower response.

Transfer Functions of OP AMP Circuits

Example 12-2: Find the transfer function of a practical differentiator- active high pass filter with definite low frequency cutoff.



Solution Method: Take Laplace transform of components and treat them like impedances.

For capacitor

$$\mathcal{L}\left[v_C(t) = \frac{1}{C} \cdot \int i_C(t)\,dt\right] = V_C(s) = \frac{1}{C \cdot s} \cdot I_C(s)$$

$$\frac{V_C(s)}{I_C(s)} = \frac{1}{C \cdot s}$$

General gain formula

$$A_v(s) = \frac{-V_o(s)}{V_i(s)} = \frac{-Z_f(s)}{Z_i(s)}$$

Define Z's

$$Z_i(s) = R_i + \frac{1}{C \cdot s}$$

$$Z_f(s) = R_f$$

Transfer Functions of OP AMP Circuits

Example 12-2 Solution (2)

$$A_v(s) = \frac{-Z_f(s)}{Z_i(s)} = \frac{-R_f}{R_i + \frac{1}{C \cdot s}}$$

Substitute into gain formula

$$A_v(s) = \left[\frac{C \cdot s}{C \cdot s}\right] \cdot \left[\frac{-R_f}{R_i + \frac{1}{C \cdot s}}\right]$$

Simplify ratio

$$A_v(s) = \left[\frac{-R_f \cdot C \cdot s}{C \cdot s \cdot (R_i + \frac{1}{C \cdot s})}\right] = \left[\frac{-R_f \cdot C \cdot s}{R_i \cdot C \cdot s + 1}\right]$$

Transfer Function

Transfer function has 1 zero at s=0 ($R_f Cs=0$) and 1 pole at $s=-1/R_i C$ ($R_i Cs+1=0$)

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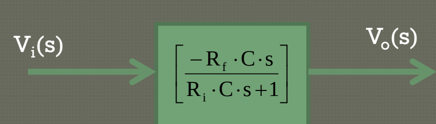
Transfer Functions of OP AMP Circuits

Block diagram equivalent of OP AMP circuit



Now consider cascaded OP AMP circuits. Similar to the constants used previously. For series connected circuits, multiply the gains (transfer functions) .
Note: do not cancel common terms from numerator and denominator.

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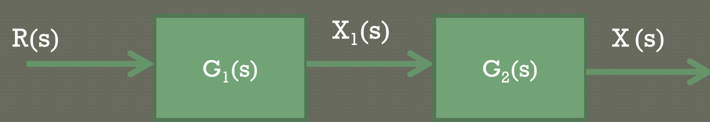
Cascaded OP AMP Circuits



Stage 1

$R(s) \cdot G_1(s) = X_1(s)$ Equation (1)

Stage 2

$X_1(s) \cdot G_2(s) = X(s)$ Equation (2)

Substitute (1) into (2) and simplify to get overall gain

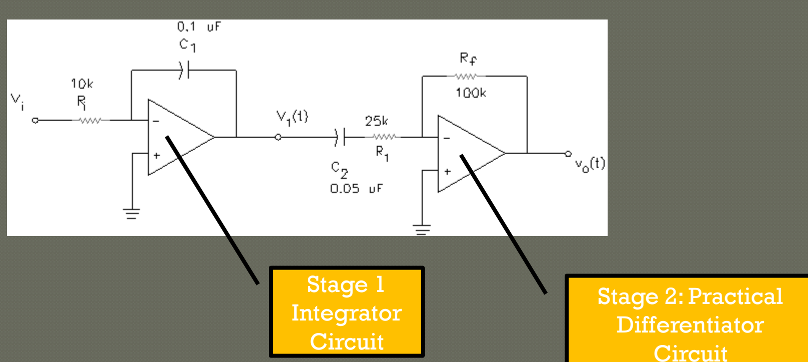
$X_1(s) = R(s) \cdot G_1(s)$

$R(s) \cdot G_1(s) \cdot G_2(s) = X(s)$

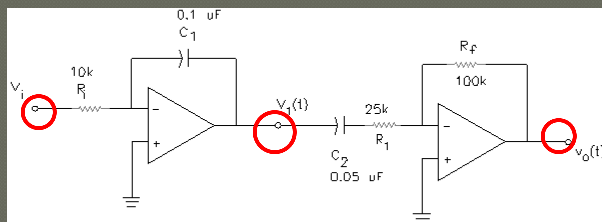
$X(s)/R(s) = G_1(s) \cdot G_2(s)$

OP AMP Example

Example 12-3: Find the transfer function of the cascaded OP AMP circuit shown. Determine the number and values of the poles and zeros of the transfer function if they exist.



Example 12-3 Solution



Solution Method
Take Laplace transform of components and use general gain formula

$$G_1(s) = A_{v1}(s) = \frac{V_1(s)}{V_i(s)}$$

Stage 1

$$G_2(s) = A_{v2}(s) = \frac{V_o(s)}{V_1(s)}$$

Stage 2

$$A_{v1}(s) = \frac{-V_1(s)}{V_i(s)} = \frac{-Z_f(s)}{Z_i(s)}$$

$$Z_i(s) = R_i$$

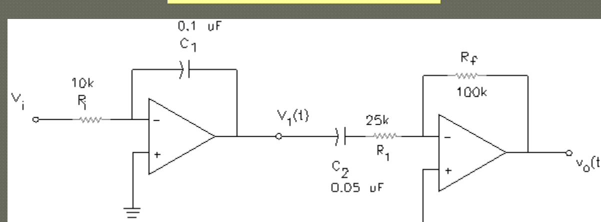
$$Z_f(s) = \frac{1}{C_1 \cdot s}$$

$$A_{v1}(s) = \left[\frac{-\left[\frac{1}{C \cdot s}\right]}{R_i}\right] = \left[\frac{C \cdot s}{C \cdot s}\right]\left[\frac{-\left[\frac{1}{C \cdot s}\right]}{R_i}\right] = \frac{-1}{R_i \cdot C \cdot s}$$

Example 12-3 Solution (2)

$G_2(s)$ was derived previously (practical differentiator)

$$G_2(s) = A_{v2}(s) = \frac{-R_f \cdot C_2 \cdot s}{R_1 \cdot C_2 \cdot s + 1}$$



$$G_1(s) \cdot G_2(s) = A_{v1}(s) \cdot A_{v2}(s) = \left[\frac{-1}{R_i \cdot C_2 \cdot s}\right]\left[\frac{-R_f \cdot C_2 \cdot s}{R_1 \cdot C_2 \cdot s + 1}\right]$$

Negative signs cancel

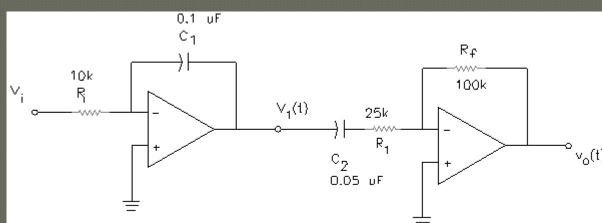
Example 12-3 Solution (3)

Overall transfer function

$$\frac{V_0(s)}{V_i(s)} = A_{v1}(s)\cdot A_{v2}(s) = \frac{R_f\cdot C_2\cdot s}{(R_i\cdot C_2\cdot s)(R_1\cdot C_2\cdot s+1)}$$

Simplified form

Plug in given values for the component symbols and compute parameters



$R_f\cdot C_2$ = 100 kΩ ·0.05 μf
$R_f\cdot C_2$ =0.005

$R_1\cdot C_2$ = 25 kΩ ·0.05 μf
$R_1\cdot C_2$ =0.001

$R_i\cdot C_1$ = 10 kΩ ·0.1 μf
$R_i\cdot C_1$ =0.001

Example 12-3 Solution (4)

Final transfer function- all values included

Function has 1 zero and 2 poles

$$\frac{V_0(s)}{V_i(s)} = \frac{0.005\cdot s}{(0.001\cdot s)(0.001\cdot s+1)}$$

Zero at s=0
0.005s=0

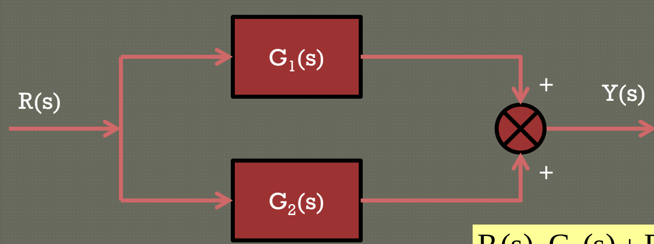
Pole at s=0
0.001s=0

Pole at s=-1/0.001=-1000
0.001s+1=0

$V_i(s)$ → $\frac{0.005\cdot s}{(0.001\cdot s)(0.001\cdot s+1)}$ → $V_o(s)$

Parallel Blocks

Overall transfer function is the algebraic sum of the signs entering summing point



$$R(s)\cdot G_1(s)+R(s)\cdot G_2(s)=Y(s)$$

$$R(s)\cdot[G_1(s)+G_2(s)]=Y(s)$$

$$\frac{Y(s)}{R(s)}=[G_1(s)+G_2(s)]$$

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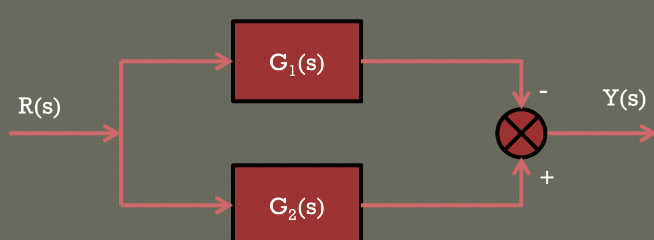
Parallel Blocks



Subtraction in Summing block

$$-R(s)\cdot G_1(s)+R(s)\cdot G_2(s)=Y(s)$$

$$R(s)\cdot[-G_1(s)+G_2(s)]=Y(s)$$

$$\frac{Y(s)}{R(s)}=[-G_1(s)+G_2(s)]$$

$$R(s)\cdot G_1(s)-R(s)\cdot G_2(s)=Y(s)$$

$$R(s)\cdot[G_1(s)-G_2(s)]=Y(s)$$

$$\frac{Y(s)}{R(s)}=[G_1(s)-G_2(s)]$$

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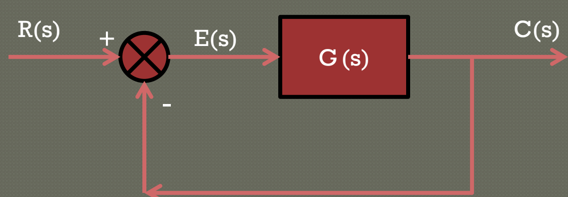
Laplace Block Simplifications

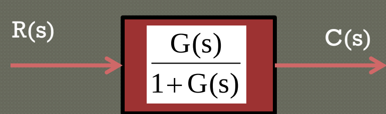
Systems with unity gain feedback



Equivalent Equation

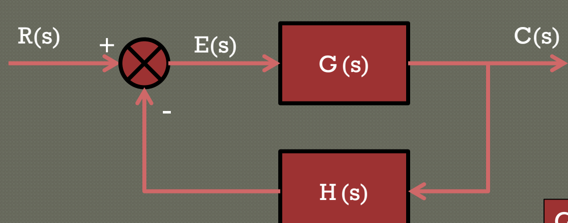
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Equivalent block



Laplace Block Simplifications

Systems with non-unity gain feedback

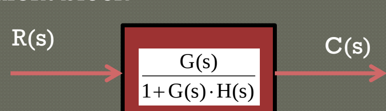


Equivalent Equation

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\cdot H(s)}$$

G(s)= forward path gain
H(s) = feedback path gain

Equivalent block



End Lesson 12: Transfer Functions In The Laplace Domain

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