Geometric diagram for relativistic addition of velocities

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A geometric diagram that allows one to visualize the Poincaré formula for relativistic addition of velocities in one dimension is presented.

I. RELATIVISTIC DIAGRAM

If James runs atop a train and his velocity with respect to the train is a, while the train has velocity b with respect to ground, what is the resulting velocity of James with respect to ground? Before 1905, we would answer as Galileo would: $a + b$. But now we know better: his velocity is

$$a \oplus b = \frac{a+b}{1+ab} \tag{1}$$

This elegant expression was discovered by Henri Poincaré[1] and constitutes the relativistic velocity-addition law in one dimension (for parallel velocities). Velocities are expressed here in natural units, in which the speed of light (unattainable for objects with mass) is 1.

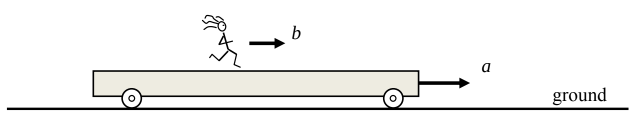

FIG. 1. Velocity a of the big platform a is measured with respect to the ground. Velocity b of the runner – with respect to the platform. What is the velocity $a \oplus b$ of the runner with respect to the ground?

It is fun to see how such a simple algebraic expression takes care of different physical situations:

James cannot exceed the speed of light:	$\frac{1}{2} \oplus \frac{1}{2} = \frac{\frac{1}{2}+\frac{1}{2}}{1+\frac{1}{2}\frac{1}{2}} = \frac{4}{5} < 1$
Even if he tries really hard:	$\frac{4}{5} \oplus \frac{5}{6} = \frac{\frac{4}{5}+\frac{5}{6}}{1+\frac{4}{5}\frac{5}{6}} = \frac{49}{50} < 1$
or flashing light while running	$v \oplus 1 = \frac{v+1}{1+v\cdot 1} = 1$
In both directions:	$v \oplus -1 = \frac{v-1}{1+v\cdot(-1)} = -1$
The desperate extreme addition:	$1 \oplus 1 = \frac{1+1}{1+1\cdot 1} = 1$

The purpose of this note is to present a simple geometric diagram (Figure 2) that allows one to visualize the Poincaré formula, and to perform the addition in a purely geometrical way.

In the figure, velocities are represented as points a and b on the real line $\mathbb{R}$. The line is embedded in $\mathbb{R}^2$ and the unit circle at the origin is added. Each velocity determines a point on the circle: namely the intersection points with the line from $(0, 1)$ through a, and with the line from $(0, -1)$ through b (or vice versa).

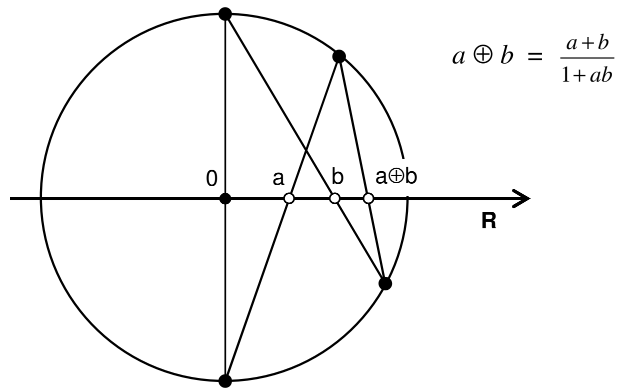

FIG. 2. Visualization of the formula for relativistic addition of velocities

Theorem. *The point on $\mathbb{R}$ that belongs to the line joining the two points just constructed represents the relativistic composition $a \oplus b$.*

Proof: Let A and B be the points on the circle determined by velocities a and b (for economy, a and b denote both points on $\mathbb{R}$ and the corresponding values). First, we find their coordinates $(x, y) \in \mathbb{R}^2$. The result, presented in the Fig. 3, may be obtained as follows.

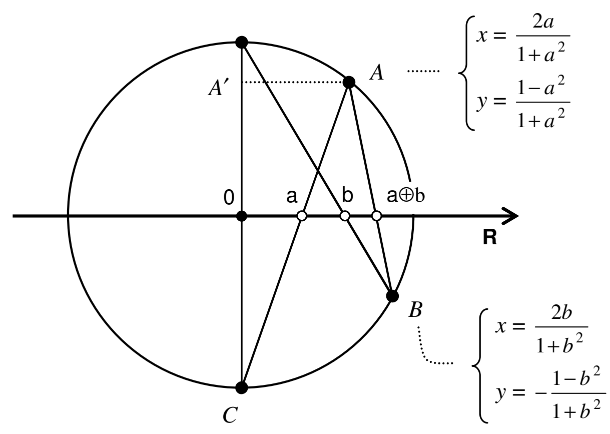

FIG. 3. Coordinates of points A and B on the circle.

Use similarity of triangles (A, A', C) and $(a, 0, C)$ to determine the coordinate value x of point A :

$$\frac{a}{1} = \frac{x}{1+y} \tag{2}$$

Squaring the above expression gives:

$$a^2 = \frac{x^2}{(1+y)^2} = \frac{1-y^2}{(1+y)^2} = \frac{(1-y)(1+y)}{(1+y)^2} = \frac{1-y}{1+y}$$

from which one readily extracts y:

$$y = \frac{1-a^2}{1+a^2}\,. \tag{3}$$

Applying this to (2) one gets

$$x = \frac{2a}{1+a^2}\,. \tag{4}$$

The coordinates for point B follow from symmetry. With these two points, A and B, in hand, one constructs the line through them:

$$y \;=\; \frac{1+ab}{a-b}x - \frac{a+b}{a-b} \qquad \text{or} \qquad (1+ab)x + (b-a)y = a+b$$

Now, by substituting $y = 0$, we get the point of intersection with the horizontal axis:

$$(x,\, y) \;=\; \left(\frac{a+b}{1+ab},\, 0\right)\,,$$

which proves the theorem. □

II. EXPERIMENTS

The diagrammatic interpretation allows one to visualize basic algebraic features of the relativistic addition formula. A few situations are shown in Fig. 4. Some extreme cases are illustrated in the bottom row. An interactive applet for more explorations may be found on the author's web page[2] (one can examine there what happens in the "forbidden" case of $1 \oplus (-1)$).

One of the speculative theories of physics is the existence of tachyons, hypothetical particles that move faster than light.[3,4] According to the theory, one of its counter-intuitive properties is that, unlike ordinary particles, a tachyon slows down as its energy increases. Tachyons are forbidden from slowing down below the speed of light, as regular matter cannot speed up above this barrier. (In both cases infinite energy would be required to reach the speed of light from either side). One can see this with the aid of our diagram (see Fig. 5): adding a small velocity to a tachyon slows it down. And subtracting — speeds it up!

We leave it to the reader to draw a diagram showing that adding two tachyonic velocities would result a subluminal speed, normal for the common matter.

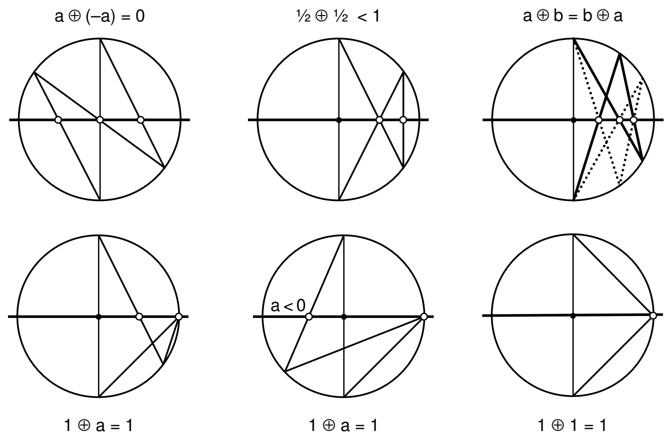

FIG. 4. Various experiments in algebra done with geometry.

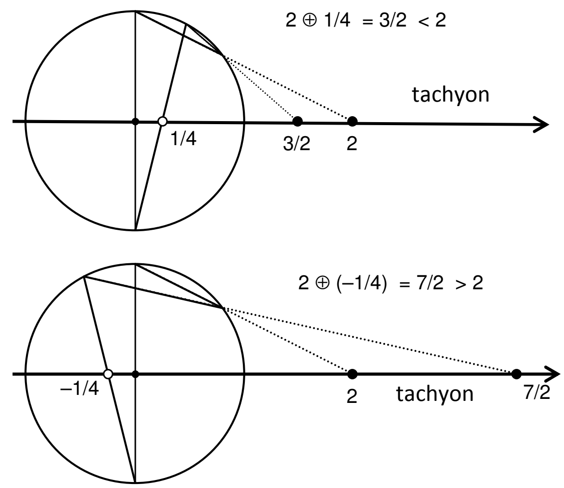

FIG. 5. Experiments with tachyons.

III. TRIGONOMETRIC TANGENT

The Poincaré formula is similar to the addition formula for tangents in regular trigonometry:

$$\tan(\alpha+\beta) \;=\; \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \qquad a \oplus b \;=\; \frac{a+b}{1-ab}$$

for economy, we use the same symbol $\oplus$ to denote the tangent addition.

Figure 6 shows how one may adjust the previous idea to the trigonometric case — namely, the regular circle must be replaced by a "hyperbolic circle". Figures 7 and 8 show a few examples. It is intriguing if not paradoxical that one needs a circle to make a construction for hyperbolic geometry and a hyperbola for the geometry of circle...

The last comment is about the analogy between the trigonometric tangent and its hyperbolic version. Actually, one may continuously deform one into the other. Deform the circle in Figure 1 into an ellipse through points

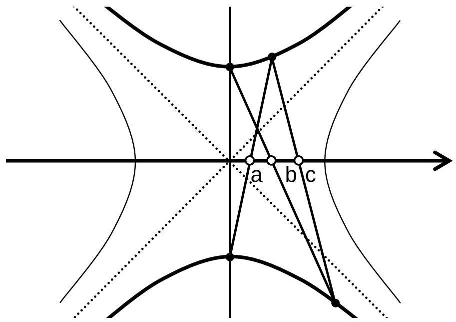

FIG. 6. Tangent-like addition

(0, ±1) and with a wider horizontal size while maintaining the geometrically defined addition. Eventually the ellipse turns into a pair of parallel lines, $y = \pm 1$ for which the geometric addition coincides with regular arithmetic addition. Increasing further the eccentricity will result in the hyperbolic case. The transformation may be executed by changing continuously the value of a from 1 to -1 in the quadratic $ax^2 + y^2 = 1$.

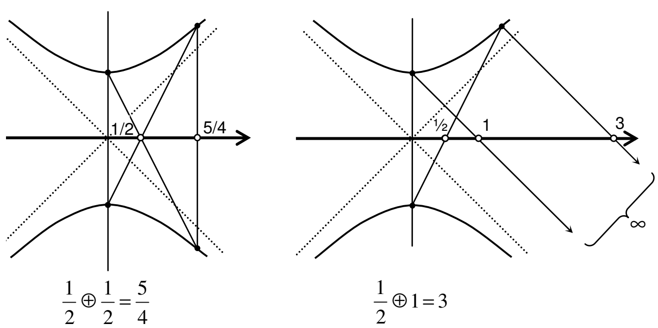

FIG. 7. Interesting cases of the tangent addition formula

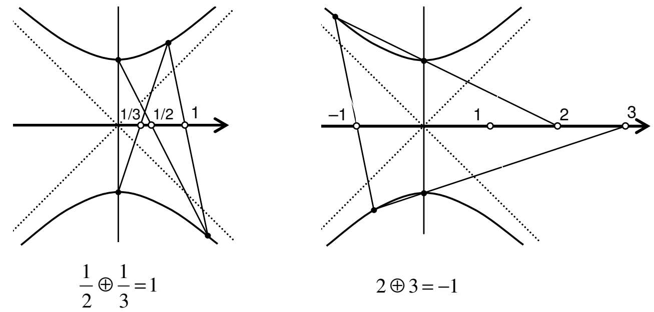

FIG. 8. More tangent additions

IV. HISTORICAL NOTE

The relativistic velocity addition formula first appears in the letter of Poincaré to Lorentz, roughly in May 1905, and was presented to the French Academy of Science in 6 June of the same year. Basic ingredients of relativity principle were presented by Poincaré at the world scientific conference in St Louis in September 1904. Poincaré also noted that the expression $d^2 = x^2 + y^2 + z^2 - c^2t^2$ defines a relativistic invariant. Although he used 4-vectors, Poincaré did not pursue the geometric consequences. It was Hermann Minkowski who followed up on this idea in 1907. Albert Einstein decisively removed the concept of ether and simplified the derivation of Lorenz' transformation rules in 1905. For more on history of relativity see Refs 5–8. For a different geometrization of relativistic velocity addition see Ref. 9.

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[1] Henri Poincaré, Letter to Hendrik Lorenz, ca. May 1905, available at http://www.univnancy2.fr/poincare/chp/text/lorentz4.xml.

[2] Jerzy Kocik, Interactive diagram for relativistic velocity addition www.math.siu.edu/Kocik/relativity/Diagram.html

[3] Olexa-Myron Bilaniuk, Vijay K. Deshpande and E. C. George Sudarshan, "Meta Relativity," Am. J. Phy. **30**, 718–723 (1962).

[4] Gerald Feinberg, "Possibility of Faster-Than-Light Particles," Phys. Rev. **159**, 1089–1105 (1967).

[5] Elie Zahar, "Poincaré's Independent Discovery of the relativity principle," Fundamenta Scientiae **4**, 147–175 (1983).

[6] Gobind Hemraj Keswani, "Origin and Concept of Relativity, Parts I, II, III," Brit. J. Phil. Sci. **15-17**, (1965-6).

[7] Abraham Pais, *Subtle is the Lord: The Science and the Life of Albert Einstein*, (New York, Oxford University Press, 1982).

[8] Roger Cerf, "Dismissing renewed attempts to deny Einstein the discovery of special relativity," Am. J. Phys. **74**, 818–824 (2006).

[9] Robert W. Brehme, "Geometrization of the relativistic velocity addition formula," Am. J. Phys. **37**, 360–363 (1969).