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Representations and Stochastic Processes on Groups of Type-H

P. FEINSILVER AND J. KOCIK

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

AND

R. SCHOTT

CRIN, Université de Nancy I, B.P. 239, 54506-Vandoeuvre-lès-Nancy, France

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We find realizations of Lie algebras of "type-H" as vectors fields. These are used in a novel approach to representing group elements as products of one-parameter subgroups (splitting formula) and for finding polynomial matrix elements of representations for the Lie group, in particular irreducible representations. Special classes of polynomials, Appell systems, and polynomial solutions to heat equations (via a Feynman–Kac type formula) are found. Additionally, we show that the Lie algebra generates Gaussian random variables. The general form of stochastic processes on the group is given, and Brownian motion is discussed in particular.

1. INTRODUCTION

In [20] Kaplan introduced a class of nilpotent Lie groups which arise naturally from the notion of composition of quadratic forms. Since this family of groups is closely related to (and includes) the Heisenberg group, they are referred as *groups of Heisenberg type* or simply of *type*-H. In [20] he showed that the standard (sub-)Laplacians on such groups admit fundamental solutions analogous to the case of the Heisenberg group [13]. The fact that type-H groups have interesting analytic as well as geometric properties sparked interest in them. The geometry and structure of type-H groups, including in particular the Heisenberg group, have been studied by Kaplan [18, 19], Kaplan and Ricci [26], Ricci [25], Korányi [23],

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Korányi and Cowling [24]. The algebraic structure particularly may be found in the works of Riehm [27].

In this paper we present new results concerning polynomial representations and stochastic processes on such groups. This work is based mainly on the studies [4, 9, 10, 11]. The approach taken here analyzes various features of type-H groups so that the detailed structures involved may be clearly seen.

From the point of view of partial differential equations, there has been much work on hypoellipticity of operators on nilpotent groups. See, for example, [3, 13, 16, 28].

Remarks. 1. Note the following summation convention: repeated *Greek* indices are considered to be summed *regardless of position.* When summations (or products) are written explicitly, Latin indices will be employed.

2. For vector variables (i.e., subscripted variables) and multi-indices bold-faced notation is used. For example, $\mathbf{X} = (X_1, ..., X_m)$. And $\mathbf{X} \cdot \mathbf{Y} = X_\mu Y_\mu = \sum X_j Y_j$, for example. Denote $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x} = x_\mu x_\mu$.

3. When the variables $\mathbf{x}$ are understood, denote $\partial/\partial x_j$ by D_j and $\mathbf{D}$ by ∇. Differentiation with respect to s or t is denoted by a dot.

2. Type-H Lie Algebras

Starting with $\mathscr{Z}$, $\mathscr{V}$ (not completely arbitrary, cf. [20]) vector spaces over $\mathbf{R}$, the corresponding (two-step) nilpotent Lie algebra $\mathscr{N} = \mathscr{V} \oplus \mathscr{Z}$ (the symbol $\oplus$ here denoting the orthogonal direct sum) is determined by a mapping $j\colon \mathscr{Z} \to \operatorname{End} \mathscr{V}$ with the properties, for $Y \in \mathscr{Z}$,

$$\begin{aligned} |j(Y)X| &= |Y|\,|X| \\ j(Y)^2 &= -|Y|^2 I \\ \langle Y, [X, X'] \rangle &= \langle j(Y)X, X' \rangle, \end{aligned} \tag{2.1}$$

where $\langle\ ,\ \rangle$ denotes a given inner product on $\mathscr{N}$. The space $\mathscr{Z}$ is the center of the algebra. Note that $j|_{\mathscr{V}} = 0$.

Now fix orthonormal bases $\{X_k\}$, $\{z_k\}$ of $\mathscr{V}$ and $\mathscr{Z}$, respectively. Let

$$J_k = j(z_k). \tag{2.2}$$

Denote

$$\begin{aligned} \dim \mathscr{V} &= m \\ \dim \mathscr{Z} &= n. \end{aligned} \tag{2.3}$$

Define the matrix elements

$$J_k^{ij} = \langle J_k X_j, X_i \rangle. \tag{2.4}$$

For each non-zero $z \in \mathscr{Z}$, $j(z)$ is surjective. For $z = c_\lambda z_\lambda \in \mathscr{Z}$, $j(c_\lambda z_\lambda) = c_\lambda J_\lambda$. Each J_k is an isometry and $J_k^2 = -I$; i.e., J_k is skew-symmetric. Recall the Clifford algebra relations [20]

$$J_k J_l + J_l J_k = -2\delta_{kl} I. \tag{2.5}$$

The following useful proposition is immediate.

2.1. PROPOSITION. *For any scalars* $\mathbf{c} = \{c_j\}$,

$$\begin{aligned} c_\lambda J_\lambda^{i\rho} c_\mu J_\mu^{j\rho} &= |\mathbf{c}|^2 \, \delta_{ij} \\ c_\lambda J_\lambda^{i\rho} c_\mu J_\mu^{\rho j} &= -|\mathbf{c}|^2 \, \delta_{ij}. \end{aligned} \tag{2.6}$$

We now find the basic commutation relations. First, calculate

$$\langle z_r, [X_k, X_l] \rangle = \langle J_r X_k, X_l \rangle = J_r^{lk}. \tag{2.7}$$

That is, $[X_k, X_l] = z_\mu J_\mu^{lk}$. Denoting this by $z \cdot J^{lk}$ (here we avoid boldface), thus

$$[X_k, X_l] = z \cdot J^{lk}. \tag{2.8}$$

3. DUAL REPRESENTATION

Using coordinates of the second kind we can dualize the action of the elements of the algebra, giving realizations as differential operators (vector fields). See [9, pp. 174–176]. Here we use the well-known relation

$$e^X Y e^{-X} = e^{\operatorname{ad} X} Y. \tag{3.1}$$

Define the group element $g(\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n)$ by

$$g = e^{\alpha_1 X_1} \cdots e^{\alpha_m X_m} e^{\beta_1 z_1} \cdots e^{\beta_n z_n}. \tag{3.2}$$

The actions of left and right multiplication dualized to vector fields acting on functions of the variables $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$ are given by

$$\begin{aligned} X_j g = X_j^\dagger g &= e^{\alpha_1 X_1} \cdots X_j e^{\alpha_j X_j} \cdots e^{\beta_n z_n} + \text{commutation terms}, \\ g X_j = X_j^* g &= e^{\alpha_1 X_1} \cdots e^{\alpha_j X_j} X_j \cdots e^{\beta_n z_n} + \text{commutation terms}, \end{aligned} \tag{3.3}$$

and are given similarly for $z_k^\dagger$, z_k^*. Note that the map $X \to X^*$ is an algebra, and hence a Lie homomorphism, while the $\dagger$ map is an anti-homomorphism.

3.1. PROPOSITION. *For the action on the left,*

$$X_j^\dagger = \partial/\partial\alpha_j + \sum_{i<j} \alpha_i J_\mu^{ij}\, \partial/\partial\beta_\mu$$
$$z_k^\dagger = \partial/\partial\beta_k. \tag{3.4}$$

And for the action on the right

$$X_j^* = \partial/\partial\alpha_j + \sum_{i>j} \alpha_i J_\mu^{ji}\, \partial/\partial\beta_\mu$$
$$z_k^* = \partial/\partial\beta_k. \tag{3.5}$$

Proof. A sample computation for each case suffices to illustrate the structure, cf. (3.1),

$$X_3 e^{\alpha_2 X_2} = e^{\alpha_2 X_2} e^{-\alpha_2 X_2} X_3 e^{\alpha_2 X_2} = e^{\alpha_2 X_2}(X_3 - \alpha_2 z \cdot J^{32})$$
$$= e^{\alpha_2 X_2}(X_3 + \alpha_2 z \cdot J^{23}) \tag{3.6}$$
$$e^{\alpha_3 X_3} X_2 = (X_2 + \alpha_3 z \cdot J^{23})\, e^{\alpha_3 X_3}. \quad ∎ \tag{3.7}$$

The adjoint representation, denoted by X', corresponds to the action by conjugation:

$$e^{X'} g = e^{\operatorname{ad} X} g = e^X g e^{-X} = e^{X^\dagger - X^*} g. \tag{3.8}$$

3.2. PROPOSITION. *The adjoint representation is given by*

$$X_j' = \alpha_\lambda J_\mu^{\lambda j}\, \partial/\partial\beta_\mu. \tag{3.9}$$

Proof. This follows directly from (3.4), (3.5), and the relation $X_j' = X_j^\dagger - X_j^*$. ∎

Next, the dual representation is used to derive a general splitting formula for Lie groups. This shows another important aspect of the dual representations.

4. SPLITTING FORMULA

The technique given here is applicable to general (local) Lie groups.

Let $\{\xi_1, ..., \xi_N\}$ be a basis for a Lie algebra. Define group elements, corresponding to coordinates of the second kind,

$$g(\alpha_1, ..., \alpha_N) = e^{\alpha_1 \xi_1} \cdots e^{\alpha_N \xi_N}$$

and, with $X = A_\mu \xi_\mu$ denoting an element of the Lie algebra, the corresponding group element, in terms of coordinates of the first kind, $\{A_j\}$, is e^X. Let $\{\xi_j^*\}$, $\{\xi_j^\dagger\}$, denote respectively the right and left dual representations (see previous section):

$$\begin{aligned} g(\alpha_1, ..., \alpha_N)\, \xi_j &= \xi_j^* g(\alpha_1, ..., \alpha_N) \\ \xi_j\, g(\alpha_1, ..., \alpha_N) &= \xi_j^\dagger\, g(\alpha_1, ..., \alpha_N). \end{aligned} \tag{4.1}$$

The $\{\xi_j^*\}$ and $\{\xi_j^\dagger\}$ are vector fields acting on functions of the $\{\alpha_j\}$. With α denoting the variables $(\alpha_1, ..., \alpha_N)$, define the matrices $\pi_{jk}^*(\alpha)$, $\pi_{jk}^\dagger(\alpha)$ by

$$\begin{aligned} \xi_j^* &= \pi_{j\mu}^*(\alpha)\, \partial/\partial\alpha_\mu \\ \xi_j^\dagger &= \pi_{j\mu}^\dagger(\alpha)\, \partial/\partial\alpha_\mu. \end{aligned} \tag{4.2}$$

The notation $\tilde{\xi}_j$ denotes *either* the left or the right representation with the corresponding relation

$$\tilde{\xi}_j = \tilde{\pi}_{j\mu}(\alpha)\, \partial/\partial\alpha_\mu. \tag{4.3}$$

4.1. LEMMA. *Let $X = A_\mu \xi_\mu$. For the one-parameter subgroup generated by X, let*

$$e^{sX} = e^{\alpha_1(s)\,\xi_1} \cdots e^{\alpha_N(s)\,\xi_N}. \tag{4.4}$$

Then the coordinates $\{\alpha_j(s)\}$ are given by the flow generated by $A_\lambda \tilde{\pi}_{\lambda\mu}(\alpha)\, \partial/\partial\alpha_\mu$,

$$\dot{\alpha} = A\tilde{\pi}(\alpha), \tag{4.5}$$

the dot denoting differentiation with respect to s; i.e.,

$$\dot{\alpha}_k(s) = A_\mu \tilde{\pi}_{\mu k}(\alpha), \tag{4.6}$$

with initial conditions $\alpha_k(0) = 0$.

Proof. Differentiating both sides of (4.4) with respect to s,

$$Xe^{sX} = \sum_{j=1}^{N} \left(\prod_{i=1}^{j-1} e^{\alpha_i(s)\,\xi_i} \right) \dot{\alpha}_j(s)\, \xi_j e^{\alpha_j(s)\,\xi_j} \left(\prod_{i=j+1}^{N} e^{\alpha_i(s)\,\xi_i} \right), \tag{4.7}$$

where the indicated products are ordered according to increasing i. For the left-hand side,

$$\begin{aligned} Xe^{sX} &= A_\mu \xi_\mu e^{sX} = A_\mu \xi_\mu\, g(\alpha(s)) \\ &= A_\mu \xi_\mu^\dagger\, g(\alpha(s)). \end{aligned} \tag{4.8}$$

And for the left representation, the result follows by the definition of $\pi^{\dagger}_{jk}(\alpha)$. For the right representation, bring down X on the right side, $e^{sX}X$, in (4.7). ∎

For two-step nilpotent groups, it is rather easy to derive the splitting rules using Campbell–Baker–Hausdorff formulas. Our general approach illustrates as well aspects of the dual representations. Another point of interest is that, as will be seen in the next section, the splitting formula is important for the connections with probability theory as it enables one to calculate the distribution of X when it can be interpreted as a random variable.

4.2. THEOREM (The Splitting Rule).

$$e^{a_\mu X_\mu} = \prod_1^m e^{a_j X_j} \cdot e^{(1/2) z \cdot J^{\rho\sigma} a_\rho a_\sigma \theta(\rho,\sigma)}, \tag{4.9}$$

where $\theta(\rho, \sigma)$ equals 1 if $\rho < \sigma$, and zero otherwise.

Proof. Let $Y = a_\mu X_\mu$. By Lemma 4.1,

$$e^{sY} = e^{\alpha_1(s) X_1} \cdots e^{\alpha_m(s) X_m} e^{\beta_1(s) z_1} \cdots e^{\beta_n(s) z_n},$$

where

$$\dot{\alpha}_k = a_k$$
$$\dot{\beta}_k = a_\mu \alpha_\lambda J_k^{\lambda\mu} \theta(\lambda, \mu)$$

via Proposition 3.1, for the left representation, say. Thus,

$$\alpha_k(s) = a_k s, \qquad \beta_k(s) = (s^2/2)\, a_\mu a_\lambda J_k^{\lambda\mu} \theta(\lambda, \mu).$$

Evaluating at $s = 1$ gives the result. ∎

For the groups laws, write

$$g(a) = e^{a_\mu X_\mu}, \qquad g(\alpha, \beta) = e^{\alpha_1 \xi_1} \cdots e^{\alpha_N \xi_N} e^{\beta_1 z_1} \cdots e^{\beta_n z_n}, \tag{4.10}$$

and similarly for $g(A)$, $g(A, B)$. Then the dual representation is used to calculate the group law $g(\alpha, \beta)\, g(A, B)$. Since the $\{z_k\}$ generate a central Abelian subgroup, we just give the result for $g(\alpha, 0)\, g(A, 0)$. And for $g(a)\, g(A)$ apply the splitting formula in conjunction with the result for $g(\alpha, \beta)\, g(A, B)$.

4.3. PROPOSITION (Group Laws). 1. *In terms of coordinates of the first kind,*

$$e^{a_\mu X_\mu} e^{A_\mu X_\mu} = e^{(a_\mu + A_\mu) X_\mu} e^{(1/2) z \cdot J^{\mu\lambda} a_\lambda A_\mu}. \tag{4.11}$$

2. *In terms of coordinates of the second kind*

$$\prod_j e^{\alpha_j X_j} \prod_k e^{A_k X_k} = \prod_j e^{(\alpha_j + A_j) X_j} \cdot e^{\alpha_\sigma A_\rho z \cdot J^{\rho\sigma}\theta(\rho,\sigma)}. \tag{4.12}$$

Proof. For part 2, using the right dual representation,

$$g(\alpha, \beta)\, g(A, 0) = e^{A_1 \xi_1^*} \cdots e^{A_N \xi_N^*} g(\alpha, \beta). \tag{4.13}$$

And the result follows from Proposition 3.1, after dropping the terms $e^{\beta_1 z_1} \cdots e^{\beta_n z_n}$. For the product $g(a)\, g(A)$, use the splitting rule as indicated in the above remarks. ▮

5. Random Variables and Stochastic Processes

The notion of a random variable is nowadays—the theory of "quantum probability"—commonly extended to self-adjoint operators, with the usual examples thought of as multiplication operators. That is, one takes the point of view common in quantum theory where the observable quantities are self-adjoint operators and functions are thought of as multiplication operators; see, e.g., [1, 4]. For type-H algebras one needs an inner product with respect to which adjoints can be defined. This can be readily done given a representation similar to the usual Schrödinger representation (alternatively, the boson Fock space) for the Heisenberg algebra, cf. the Bargmann representation [12, also see 14]. Namely, take a cyclic vector Ω such that $z_j \Omega = c_j \Omega$ for some given scalars c_j. Then the Lie bracket on $\mathcal{N}$ determines a skew-symmetric form B on $\mathcal{N} \times \mathcal{N}$ via $B(X, Y) = [X, Y]\Omega$, $X, Y \in \mathcal{N}$. Observe that B is non-degenerate. With $B(X_j, X_i) = c_\mu J^{ij}_\mu$, for any vector ξ, by Eqs. (2.5) and (2.1):

$$\begin{aligned} \langle c_\lambda J_\lambda \xi, c_\mu J_\mu \xi \rangle &= c_\mu c_\mu \langle J_\mu \xi, J_\mu \xi \rangle \\ &= |c|^2\, |\xi|^2. \end{aligned} \tag{5.1}$$

By Darboux' lemma, make an orthogonal change of variables such that B takes the form of a direct sum of 2×2 matrices of the form $\left(\begin{smallmatrix} 0 & \mu \\ -\mu & 0 \end{smallmatrix}\right)$. Arrange this so that the μ's are all positive. Associated to each block, define pairs of elements X_i, Y_i such that $B(Y, X) = \mu$. This yields a decomposition into "baby Heisenberg" algebras. The representation space consists of all polynomials in the (commuting) variables X_i acting on Ω. Defining $Y_j \Omega = 0$, $\forall j$, the inner product is defined so that multiplication by X_i is adjoint to the action of Y_i, for all i. (For a Hilbert space, one can now take the closure.) Define the "vacuum expectation values" $\langle Q \rangle_\Omega = \langle Q\Omega, \Omega \rangle$ for operators Q on this Fock-type space.

5.1. PROPOSITION. *The elements $X_i + Y_i$ form a family of independent Gaussian random variables.*

Proof. The elements $\Xi = a_\rho(X_\rho + Y_\rho)$ can be considered as random variables. We compute the characteristic function, i.e., the Fourier transform of the corresponding (spectral) measure,

$$\langle e^{s\Xi} \rangle_\Omega = e^{(1/2)\, s^2 a_\rho^2 \mu_\rho}, \tag{5.2}$$

using the splitting rule (Proposition 4.2) and the fact that all X and Y variables drop out—the Y's give zero on Ω and the X's may be moved across to become Y's. Since this is quadratic in s, the corresponding random variable Ξ is Gaussian. In other words, the elements $X_i + Y_i$ are independent (jointly) Gaussian random variables. ∎

In general, without changing the basis, one gets for the characteristic function of $a_\rho X_\rho$, from the splitting rule,

$$e^{(1/2)\, s^2 c_\mu J_\mu^{\rho\sigma} a_\rho a_\sigma \theta(\rho, \sigma)} \tag{5.3}$$

and to get a legitimate random variable one needs the quantity in the exponent to be positive. This positivity condition is the requirement for the interpretation of the X_i as a Gaussian family, without the need for a change-of-basis.

Next, consider the form a stochastic process takes when mapped via the exponential map to type-H groups. See [14, 10, 11] for a detailed discussion. We consider the process as built up successively in steps: let $\alpha_j^{(N)}$ denote the increment of the jth component, corresponding to X_j at the Nth (time-)step and let $A_k^{(N-1)}$ denote the kth component of the process up to (including) step $N-1$. Then, by the group law (Prop. 4.3),

$$\prod_j e^{\alpha_j^{(N)} X_j} \prod_k e^{A_k^{(N-1)} X_k} = \prod_k e^{A_k^{(N)} X_k} \cdot e^{A_\rho^{(N-1)} \alpha_\sigma^{(N)} z \cdot J^{\rho\sigma} \theta(\rho, \sigma)}. \tag{5.4}$$

5.2. THEOREM. *For a continuous time process with components $w_k(t)$ on $\mathbf{R}^d$ the canonical components (i.e., in terms of coordinates of the first kind) of the corresponding process on the group are given by*

$$\left(w_1, ..., w_m, \tfrac{1}{2} \int_0^t J_1^{\alpha\beta} w_\alpha \, dw_\beta, ..., \tfrac{1}{2} \int_0^t J_n^{\alpha\beta} w_\alpha \, dw_\beta \right) \tag{5.5}$$

Proof. In the limit, (5.4) yields

$$\prod e^{w_k(t)\, X_k} \cdot e^{\int_0^t w_\rho(s)\, dw_\sigma(s) z \cdot J^{\rho\sigma} \theta(\rho, \sigma)}. \tag{5.6}$$

Using the splitting rule

$$\prod e^{w_k X_k} = e^{w_\mu X_\mu} e^{-(1/2) z \cdot J^{\rho\sigma} w_\rho w_\sigma \theta(\rho, \sigma)} \tag{5.7}$$

with $d(w_\rho w_\sigma) = w_\rho \, dw_\sigma + w_\sigma \, dw_\rho$,

$$e^{w_\mu X_\mu} e^{(1/2) z \cdot J^{\rho\sigma} \theta(\rho, \sigma) \int_0^t (w_\rho \, dw_\sigma - w_\sigma \, dw_\rho)} = e^{w_\mu X_\mu + (1/2) \int_0^t z \cdot J^{\alpha\beta} w_\alpha \, dw_\beta}. \quad \blacksquare \tag{5.8}$$

In particular, for standard Brownian motion $\mathbf{w}(t)$ on $\mathbf{R}^m$, here using $\langle \ \rangle$ to denote the expected value, for g given by (3.2), with $\beta_k = 0$, $1 \leqslant k \leqslant n$.

5.3. THEOREM. *For Brownian motion we have the relation*

$$e^{tH} = \langle g(w_1, \ldots, w_m) \, e^{\int_0^t w_\rho \, dw_\sigma z \cdot J^{\rho\sigma} \theta(\rho, \sigma)} \rangle, \tag{5.9}$$

where $H = \frac{1}{2} \sum X_j^2$ is the standard Laplacian on the group.

Proof. We just indicate the formal idea. Namely, look at an increment, cf. Eq. (3.2), $g(\alpha_1^{(N)}, \ldots, \alpha_m^{(N)})$, in (5.4). Since this is in factored form, dropping the (N)'s for convenience,

$$\langle g(\alpha_1, \ldots, \alpha_m) \rangle = e^{(1/2N) X_1^2} \cdots e^{(1/2N) X_m^2}, \tag{5.10}$$

for α_j independent Gaussian variables with mean zero and variance $1/N$. Taking products and using the Trotter product formula yields the result in the limit. $\blacksquare$

See [10, 11] for more details.

Remark. Theorem 5.3 gives the explicit form of the heat flow. Formula (5.9) can also be deduced from the study done by Gaveau of the free step 2 nilpotent Lie groups and corresponds to the definition of the solution of the stochastic Stratonovich equation driven by vector fields such as those given in Proposition 3.1.

Next, we see the form of Hamilton's equations for these groups.

6. HAMILTON'S EQUATIONS

We would like to find the equations for a flow of the form $e^{tH} \xi e^{-tH}$, where H is a function on the algebra $\mathcal{N}$, e.g., a polynomial in the X variables (or a formal power series) or a combination of exponentials e^{aX}. That is, reduce to consideration of a Hamiltonian of the form $H(X_1, \ldots, X_m) = H_1(X_1) \cdots H_m(X_m)$. Then, as in the calculation of the dual representations,

$$\begin{aligned}[H, X_i] &= (-\text{ad } X_i)(H) \\ &= -\sum H_1(X_1) \cdots [X_i, H(X_j)] \cdots H(X_m) \\ &= \sum_j [X_j, X_i] \frac{\partial H}{\partial X_j}.\end{aligned} \tag{6.1}$$

Compare the calculation (recall Eq. (3.8))

$$\left.\frac{\partial}{\partial t}\right|_0 e^{tX_i} g e^{-tX_i} = [X_i, g] = X_i' g = [X_i, X_\mu] \frac{\partial g}{\partial X_\mu}. \tag{6.2}$$

Thus,

6.1. THEOREM. *Hamilton's equations are given by*

$$\begin{aligned}\dot{X}_i &= \sum_j [X_j, X_i] \frac{\partial H}{\partial X_j} \qquad \text{or} \qquad & \dot{X}_i &= z \cdot J^{i\mu} \frac{\partial H}{\partial X_\mu} \\ \dot{z}_i &= 0 & \dot{z}_i &= 0.\end{aligned} \tag{6.3}$$

These can be written compactly in the form

$$\begin{aligned}\dot{\mathbf{X}} &= z \cdot J\nabla H \\ \dot{\mathbf{z}} &= 0.\end{aligned} \tag{6.4}$$

The skew-symmetric form $z \cdot J$ gives a Poisson structure—and thus one has "classical" as well as "quantum" mechanics associated to these groups.

In the next section, explicit representations of groups of type-H on spaces of polynomials are given.

7. MATRIX ELEMENTS AND REPRESENTATIONS

We use the technique of dual representations (Section 3 above) to compute matrix elements for the action by left multiplication of the group elements (3.2) on the corresponding universal enveloping algebra. First the "general case" is considered, with the action on the full algebra. Then we present the representations where the elements of the center $\mathscr{Z}$ act as scalars, including the irreducible representations. See also [14]. For a full discussion of the technique in general, see [9].

Remarks. We employ multi-index notation. Here $\mathbf{l}$, $\mathbf{n}$, e.g., denote multi-indices: $\mathbf{l} = (l_1, ..., l_d)$, e.g., $\mathbf{l}! = l_1! \cdots l_d!$, $|\mathbf{l}| = l_1 + \cdots + l_d$, $\alpha^{\mathbf{l}} = \alpha_1^{l_1} \cdots \alpha_d^{l_d}$. The index j will always denote a *single* index. Greek indices correspond appropriately to their Latin counterparts. Typically λ, μ will be *single*, while ρ will be used for a running multi-index. For d-dimensional multi-indices, $\mathbf{e}_j$ will denote the standard basis elements in $\mathbf{N}^d$. Recall, Eqs. (2.3), $n = \dim \mathscr{X}$.

7.1. *General Rrpresentation*

The basis for the enveloping algebra is taken in the form $\psi_{\mathbf{n}l} = \mathbf{X}^{\mathbf{n}}\mathbf{z}^{l}$. The action of the group element g (3.2) gives matrix elements according to

$$g\psi_{\mathbf{n}l} = \sum \mathscr{M}_{\mathbf{n}l}^{\mathbf{NL}} \psi_{\mathbf{n}+\mathbf{N}l+\mathbf{L}}, \tag{7.1.1}$$

where the capitals indicate the effective shift in the corresponding indices. Following the discussion in [9, pp. 174–176], use the right dual action to move a ψ term across and thus write the matrix elements in the form

$$\mathscr{M}_{\mathbf{n}l}^{\mathbf{NL}} = \mathbf{X}^{*\mathbf{n}}\mathbf{z}^{*l} \frac{\alpha^{\mathbf{n}+\mathbf{N}}\beta^{l+\mathbf{L}}}{(\mathbf{n}+\mathbf{N})!\,(l+\mathbf{L})!}. \tag{7.1.2}$$

7.1.1. THEOREM. *The matrix elements for the general representation of the group acting on the enveloping algebra are given by*

$$\mathscr{M}_{\mathbf{n}l}^{\mathbf{NL}} = \frac{\alpha^{\mathbf{N}}\beta^{\mathbf{L}}}{\mathbf{N}!\,\mathbf{L}!} \sum_{\rho = |\sigma_j|} \prod \frac{(-\mathbf{n}_j)_{\rho_j}(-\mathbf{L}_k)_{|\sigma_k|}}{(1+\mathbf{N}_j)_{\rho_j}\,\sigma_{kj}!} \alpha_j^{\rho_j} \left(\frac{\alpha_{j+\lambda}J_k^{jj+\lambda}}{\beta_k}\right)^{\sigma_{kj}}, \tag{7.1.3}$$

where the implied sum over λ is for $\lambda > 0$, and σ_k, σ_j each have components σ_{kj}.

Proof. As in (3.5), here $\lambda > 0$,

$$X_j^* = \partial/\partial\alpha_j + \alpha_{j+\lambda}J_\mu^{jj+\lambda}\,\partial/\partial\beta_\mu. \tag{7.1.4}$$

Thus, the expansions

$$(X_j^*)^{n_j} = \binom{n_j}{\rho_j}(\partial/\partial\alpha_j)^{n_j-\rho_j}(\alpha_{j+\lambda}J_\mu^{jj+\lambda}\,\partial/\partial\beta_\mu)^{\rho_j}. \tag{7.1.5}$$

This gives, since $z_j^* = \partial/\partial\beta_j$,

$$\begin{aligned}
\mathscr{M}_{\mathbf{n}l}^{\mathbf{NL}} &= \prod \binom{n_j}{\rho_j}(\partial/\partial\alpha_j)^{n_j-\rho_j}(\alpha_{j+\lambda}J_\mu^{jj+\lambda}\,\partial/\partial\beta_\mu)^{\rho_j}\frac{\alpha^{\mathbf{n}+\mathbf{N}}\beta^{\mathbf{L}}}{(\mathbf{n}+\mathbf{N})!\,\mathbf{L}!} \\
&= \prod \binom{n_j}{\rho_j}(\partial/\partial\alpha_j)^{n_j-\rho_j}\binom{\rho_j}{\sigma_{1j}, ..., \sigma_{nj}} \\
&\quad \times (\alpha_{j+\lambda}J_k^{jj+\lambda})^{\sigma_{kj}}(\partial/\partial\beta_k)^{\sigma_{kj}}\frac{\alpha^{\mathbf{n}+\mathbf{N}}\beta^{\mathbf{L}}}{(\mathbf{n}+\mathbf{N})!\,\mathbf{L}!} \\
&= \prod \binom{n_j}{\rho_j}\binom{\rho_j}{\sigma_{1j}, ..., \sigma_{nj}}\frac{\alpha^{\mathbf{N}+\rho}\beta^{\mathbf{L}}}{(\mathbf{N}+\rho)!\,\mathbf{L}!}\frac{\mathbf{L}!}{(\mathbf{L}-\sigma)!}\left(\frac{\alpha_{j+\lambda}J_k^{jj+\lambda}}{\beta_k}\right)^{\sigma_{kj}},
\end{aligned} \tag{7.1.6}$$

where the components of the index σ are $\sigma_k = \sum_j \sigma_{kj}$. Note that $\rho_j = \sum_k \sigma_{kj}$. ∎

As mentioned in [5], where the Heisenberg group is discussed, these are generalized Appell functions.

7.2. *Irreducible Representations*

For irreducible representations (cf. [14], also see [9]), the action of the z variables on a cyclic vector is given by $z_j\Omega = h_j\Omega$, for some scalars h_j. The basis reduces to $\mathbf{X}^{\mathbf{n}}\Omega$. We have the action of g

$$g\mathbf{X}^{\mathbf{n}}\Omega = e^{\beta \cdot \mathbf{h}} e^{\alpha \cdot \mathbf{X}} \mathbf{X}^{\mathbf{n}}\Omega. \tag{7.2.1}$$

7.2.1. THEOREM. *The matrix elements for irreducible representations of groups of type-*H *have the form*

$$\mathcal{M}_{\mathbf{n}}^{\mathbf{N}} = \frac{\alpha^{\mathbf{N}}}{\mathbf{N}!} e^{\beta \cdot \mathbf{h}} \prod_j {}_1F_1\left(\begin{matrix} -n_j \\ 1+N_j \end{matrix} \middle| -\alpha_j(\alpha J\mathbf{h})_j\right), \tag{7.2.2}$$

where $(\alpha J\mathbf{h})_j = \alpha_{j+\lambda} J_\mu^{jj+\lambda} h_\mu$.

Proof. The matrix elements are

$$\begin{aligned}\mathcal{M}_{\mathbf{n}}^{\mathbf{N}} &= \prod_j (\partial/\partial\alpha_j + \alpha_{j+\lambda} J_\mu^{jj+\lambda}\, \partial/\partial\beta_\mu)^{n_j} e^{\beta \cdot \mathbf{h}} \frac{\alpha^{\mathbf{n}+\mathbf{N}}}{(\mathbf{n}+\mathbf{N})!} \\ &= \prod \binom{n_j}{\rho_j} (\partial/\partial\alpha_j)^{n_j-\rho_j} (\alpha_{j+\lambda} J_\mu^{jj+\lambda} h_\mu)^{\rho_j} \frac{\alpha^{\mathbf{n}+\mathbf{N}}}{(\mathbf{n}+\mathbf{N})!} e^{\beta \cdot \mathbf{h}} \\ &= \frac{\alpha^{\mathbf{N}}}{\mathbf{N}!} e^{\beta \cdot \mathbf{h}} \prod \frac{(-n_j)_{\rho_j}}{\rho_j!} \frac{(-\alpha_j(\alpha J\mathbf{h})_j)^{\rho_j}}{(N_j+1)_{\rho_j}},\end{aligned} \tag{7.2.3}$$

where $(\alpha J\mathbf{h})_j = \alpha_{j+\lambda} J_\mu^{jj+\lambda} h_\mu$. This gives result. ∎

A detailed discussion of the role of Laguerre polynomials (one-variable ${}_1F_1$'s) in analysis on the Heisenberg group is given in [2].

8. NATURAL REALIZATIONS AND POLYNOMIAL REPRESENTATIONS

Dualize (3.5), by the algebraic version of Fourier transformation,

$$\begin{aligned}\partial/\partial\alpha_j &\to x_j \\ \alpha_j &\to -D_j,\end{aligned} \tag{8.1}$$

and define the matrix K by

$$K^{ij} = z_\mu J_\mu^{ij} \theta(i, j), \tag{8.2}$$

i.e., K consists of the upper triangular part of $z \cdot J$. Then

$$X_j = x_j - K^{j\mu} D_\mu \tag{8.3}$$

or $\mathbf{X} = \mathbf{x} - K\mathbf{D}$. One immediately checks the commutation relations (2.8). Note, Eq. (3.1), that (8.3) can be written in the form

$$X_j = e^{-K^{j\mu} D_\mu D_j} x_j e^{K^{j\mu} D_\mu D_j}. \tag{8.4}$$

In the realization (8.3), since the D's act only on higher-numbered variables, the following proposition is immediate. Here Ω is given by the constant function 1.

8.1. PROPOSITION. *Consider the basis* $\zeta_{\mathbf{p}} = X_m^{p_m} \cdots X_1^{p_1} \Omega$. *In the realization* (8.3), $\zeta_{\mathbf{p}}$ *reduces to*

$$\zeta_{\mathbf{p}} = x_m^{p_m} \cdots x_1^{p_1}. \tag{8.5}$$

For example, given a function L analytic at 0, one can define the associated *Appell polynomials* (see [6–8]) by

$$\phi_{\mathbf{p}} = e^{tL} \zeta_{\mathbf{p}}, \tag{8.6}$$

with $L = L(D_1, ..., D_m)$, $L\Omega = 0$. Define

$$\mathscr{C} = e^{tL} \mathbf{X} e^{-tL}. \tag{8.7}$$

Then $\phi_{\mathbf{p}} = \mathscr{C}^{\mathbf{p}} \Omega$. And $\mathscr{C} = \mathbf{C} - K\mathbf{D}$, where the "creation operator" $\mathbf{C}$ is defined by

$$\mathbf{C} = e^{tL} \mathbf{x} e^{-tL}. \tag{8.8}$$

As for Proposition 8.1, it follows that

8.2. PROPOSITION. *The Appell polynomials satisfy*

$$\phi_{\mathbf{p}} = \mathscr{C}^{\mathbf{p}} \Omega = \mathbf{C}^{\mathbf{p}} \Omega. \tag{8.9}$$

That is, because of the particular ordering of the variables you get nothing new, which is not uninteresting in itself. Namely, these representations of the Heisenberg algebra are the same in the type-H case as well.

It is of interest to calculate the realization of the basis elements, used already in Section 7 for the irreducible representations, namely

$$\psi_{\mathbf{n}} = X_1^{n_1} \cdots X_m^{n_m} \Omega. \tag{8.10}$$

For example, for $m=2$, with $X_1=x_1+\alpha D_2$, $X_2=x_2$,

$$(x_1+\alpha D_2)^{n_1}\,x_2^{n_2}=x_1^{n_1}x_2^{n_2}{}_2F_0\left(\begin{matrix}-n_1,\ -n_2\\ \text{---}\end{matrix}\,\middle|\,\frac{\alpha}{x_1x_2}\right). \tag{8.11}$$

8.3. Lemma. *Multiplication by X_j on the basis $\psi_{\mathbf{n}}$ is given by*

$$X_j\psi_{\mathbf{n}}=\psi_{\mathbf{n}+\mathbf{e}_j}+n_\mu K^{\mu j}\psi_{\mathbf{n}-\mathbf{e}_\mu}. \tag{8.12}$$

Proof. For $k<j$, $[X_j, X_k^r]=rX_k^{r-1}K^{kj}$. The result follows as for the left action in Proposition 3.1, Eq. (3.4). ∎

In terms of $(x_1, ..., x_m)$, using the realization (8.3), set

$$\phi_{\mathbf{n}}(x_1, ..., x_m)=\psi_{\mathbf{n}}. \tag{8.13}$$

8.4. Theorem. *$\phi_{\mathbf{n}}$ given by (8.13) satisfy the recursion*

$$x_j\phi_{\mathbf{n}}=\phi_{\mathbf{n}+\mathbf{e}_j}+K^{j\mu}D_\mu\phi_{\mathbf{n}}+K^{\mu j}n_\mu\phi_{\mathbf{n}+\mathbf{e}_\mu}. \tag{8.14}$$

Proof. From (8.12) and (8.3),

$$\begin{aligned}\phi_{\mathbf{n}+\mathbf{e}_j}&=X_j\phi_{\mathbf{n}}-n_\mu K^{\mu j}\phi_{\mathbf{n}+\mathbf{e}_\mu}\\ &=(x_j-K^{j\mu}D_\mu)\,\phi_{\mathbf{n}}-n_\mu K^{\mu j}\phi_{\mathbf{n}+\mathbf{e}_\mu},\end{aligned} \tag{8.15}$$

which yields the result. ∎

Write

$$\phi_{\mathbf{n}}=(x_1+K^{1\mu}D_\mu)^{n_1}\cdots(x_{m-1}+K^{m-1m}D_m)^{n_{m-1}}\,x_m^{n_m}. \tag{8.16}$$

Expand using multinomial coefficients and get a multivariate generalized hypergeometric function. Since in the present context the general expression appears to be of purely formal interest, we omit writing it out. However, as in Section 3, cf. also [9], the detailed structure of these polynomials is certainly of interest, particularly from the viewpoint of representations of nilpotent groups in general.

Another natural realization is the use of the full matrices $z\cdot J$; see, e.g., [14, 17]. This is used in the following section. Namely, let

$$X_j=x_j+A^{j\mu}D_\mu, \tag{8.17}$$

where the matrix A is given by $A^{ij}=-\frac{1}{2}z\cdot J^{ij}$, and thus is antisymmetric. The commutation relations (2.8) are readily checked.

9. Heat Polynomials

We will discuss computation of the action of e^{tH} with $H=\frac{1}{2}\sum X_j^2$, using the results of Section 6, acting on polynomial bases. First, deduce via Theorem 6.1 the flow

$$\mathbf{X}(t)=e^{tH}\mathbf{X}e^{-tH}=(\cos t\,|\mathbf{z}|)\mathbf{X}+\frac{\sin t\,|\mathbf{z}|}{|\mathbf{z}|}\,z\cdot J\mathbf{X}, \tag{9.1}$$

which we write in the form $\mathbf{X}(t)=\kappa\mathbf{X}+\sigma z\cdot J\mathbf{X}$.

9.1. Proposition. *$\xi_i=\kappa X_i$, $\delta_i=\sigma z\cdot J^{i\mu}X_\mu$ generate a Heisenberg algebra. In fact,*

$$[\delta_i,\xi_i]=\sigma\kappa\,|\mathbf{z}|^2. \tag{9.2}$$

Proof. Recalling that, Proposition 2.1, $z\cdot J^{i\mu}z\cdot J^{i\mu}=|\mathbf{z}|^2$, calculate

$$[\sigma z\cdot J^{i\mu}X_\mu,\kappa X_i]=\sigma\kappa\,|\mathbf{z}|^2 \tag{9.3}$$

as required. ∎

Thus, $X_i(t)$ is of the form $\xi_i+\delta_i$, with δ_i, ξ_i, generating a Heisenberg algebra. Now recall the Appell–Hermite polynomials, $h_n(x,t)$, with generating function

$$e^{\alpha x+\alpha^2 s/2}=\sum_{n=0}^{\infty}\frac{\alpha^n h_n(x,s)}{n!} \tag{9.4}$$

(Hermite polynomials have $-s$ replacing s).

9.2. Lemma. *With $\kappa=\cos t\,|\mathbf{z}|$ and $\sigma=|\mathbf{z}|^{-1}\sin t\,|\mathbf{z}|$,*

$$X_i(t)^n=\sum_{k=0}^{n}\binom{n}{k}\kappa^{n-k}X_i^{n-k}h_k(\sigma z\cdot J^{i\mu}X_\mu,\sigma\kappa\,|\mathbf{z}|^2). \tag{9.5}$$

Proof. By the splitting rule in the Heisenberg case,

$$e^{\alpha X_i(t)}=e^{\alpha\xi_i}e^{\alpha\delta_i}e^{(\alpha^2/2)\sigma\kappa\,|\mathbf{z}|^2}. \tag{9.6}$$

Thus,

$$X_i(t)^n=\sum_{k=0}^{n}\binom{n}{k}\xi_i^{n-k}h_k(\delta_i,\sigma\kappa\,|\mathbf{z}|^2), \tag{9.7}$$

and the result follows. ∎

Using the realization (8.3), e.g., we can calculate *generalized Appell polynomials*, cf. (8.6), $\zeta_{\mathbf{p}}(t) = e^{tH} X_m^{p_m} \cdots X_1^{p_1} e^{-tH} \Omega$.

9.3. THEOREM. *The generalized Appell polynomials can be computed by*

$$X_m(t)^{p_m} \cdots X_1(t)^{p_1} \Omega, \tag{9.8}$$

with $X_i(t)$ given in Lemma 9.2.

With Ω such that $H\Omega = 0$ in any particular realization, we have directly $e^{tH} \zeta_{\mathbf{p}}$.

One can also use the basis $\psi_{\mathbf{n}} = X_1^{n_1} \cdots X_m^{n_m} \Omega$. This leads to a somewhat different approach. Recall the matrix elements, Theorem 7.2.1, for the action of g: $g\psi_n = \sum \mathcal{M}_{\mathbf{n}}^{\mathbf{N}} \psi_{\mathbf{n}+\mathbf{N}}$.

9.4. THEOREM. *The heat flow is given by*

$$e^{tH} \psi_{\mathbf{n}} = \sum \langle \mathcal{M}_{\mathbf{n}}^{\mathbf{N}}(\mathbf{w})\, e^{\int_0^t w_\rho\, dw_\sigma\, K^{\rho\sigma}} \rangle\, \psi_{\mathbf{n}+\mathbf{N}}, \tag{9.9}$$

where the $\mathcal{M}_{\mathbf{n}}^{\mathbf{N}}$, given by Theorem 7.2.1, *are polynomials in* $\mathbf{w}$.

Proof. From Theorem 5.3, using K from (8.2),

$$e^{tH} = \langle g(w_1, ..., w_m)\, e^{\int_0^t w_\rho\, dw_\sigma\, K^{\rho\sigma}} \rangle. \tag{9.10}$$

Applying this to $\psi_{\mathbf{n}}$,

$$e^{tH} \psi_{\mathbf{n}} = \langle g(w_1, ..., w_m)\, \psi_{\mathbf{n}} e^{\int_0^t w_\rho\, dw_\sigma\, K^{\rho\sigma}} \rangle, \tag{9.11}$$

and the result follows. ∎

Thus one is interested in the joint distribution, equivalently the characteristic function or moment-generating function,

$$\langle e^{\alpha_1 w_1 + \cdots + \alpha_m w_m} e^{\int_0^t w_\rho\, dw_\sigma\, K^{\rho\sigma}} \rangle. \tag{9.12}$$

Notice that here K is only upper triangular, in contrast to the case considered in [17], where the full anti-symmetric matrix appears. Here we follow the lines of [10], i.e., using the algebraic structure directly to calculate e^{tH}. As in [17], use the realization with the full matrix $z \cdot J$ mentioned at the end of Section 8, Eq. (8.17). I.e., $\mathbf{X} = \mathbf{x} + A\mathbf{D}$, with $A = -\frac{1}{2} z \cdot J$. (Of course a probabilistic approach to (9.9) is of interest as well.)

We have $H = \frac{1}{2} \sum X_j^2$, with $\mathbf{X} = \mathbf{x} + A\mathbf{D}$. Introduce the variables

$$\begin{aligned} \delta_i &= A^{i\mu} D_\mu \\ R_i &= \tfrac{1}{2} x_i^2 \\ L_i &= \tfrac{1}{2} \delta_i^2. \end{aligned} \tag{9.13}$$

Thus, $[\delta_i, x_j] = A^{ij}$.

9.5. LEMMA. *We have the commutation relations*

$$
\begin{aligned}
[L_k, R_j] &= (x_j\delta_k + \tfrac{1}{2}A^{kj})\,A^{kj} \\
[L_i, x_j\delta_k] &= A^{ij}\delta_i\delta_k \qquad (9.14)\\
[x_i\delta_j, R_k] &= A^{jk}x_ix_k. \quad ∎
\end{aligned}
$$

Note that $X_i = x_i + \delta_i$, with $[\delta_i, x_i] = 0$. Thus H takes the form

$$H = \tfrac{1}{2}\sum (x_i^2 + \delta_i^2) + x_\mu\delta_\mu = \sum (R_j + L_j) + x_\mu\delta_\mu. \qquad (9.15)$$

Now put

$$
\begin{aligned}
\Delta &= \sum L_j = \tfrac{1}{2}\sum (A^{j\mu}D_\mu)^2 \\
P &= \sum R_j = \tfrac{1}{2}\sum x_j^2 \qquad (9.16)\\
E &= [\Delta, P] = (x_\lambda\delta_\mu + \tfrac{1}{2}A^{\mu\lambda})\,A^{\mu\lambda},
\end{aligned}
$$

thus defining E, cf. (9.14). An *sl*(2) *algebra* is a Lie algebra with three generators, A, B, C, say, satisfying the relations

$$
\begin{aligned}
[C, A] &= B \\
[B, A] &= \lambda A \qquad (9.17)\\
[C, B] &= \lambda C
\end{aligned}
$$

for some constant λ. A *standard sl*(2) *algebra* is the Lie algebra with three generators, conventionally denoted by Δ, ρ, R, satisfying the relations

$$
\begin{aligned}
[\Delta, R] &= \rho \\
[\rho, R] &= 2R \qquad (9.18)\\
[\Delta, \rho] &= 2\Delta.
\end{aligned}
$$

9.6. LEMMA. *The variables* Δ, E, *and* P *generate an sl*(2) *algebra, with* $\lambda = \frac{1}{2}|\mathbf{z}|^2$.

Proof. Using (9.14), (9.16),

$$[\Delta, E] = \left[\sum L_j, x_\rho\delta_\sigma A^{\sigma\rho}\right] = A^{\mu\rho}\delta_\mu\delta_\sigma A^{\sigma\rho}. \qquad (9.19)$$

Recalling that $A = -\frac{1}{2}z\cdot J$, from Proposition 2.1,

$$A^{j\rho}A^{s\rho} = \tfrac{1}{4}z\cdot J^{j\rho}z\cdot J^{s\rho} = \tfrac{1}{4}|\mathbf{z}|^2\,\delta_{js} \qquad (9.20)$$

This gives $[\Delta, E] = \frac{1}{2}|\mathbf{z}|^2\,\Delta$. Similarly, $[E, P] = \frac{1}{2}|\mathbf{z}|^2\,P$. ∎

For the standard $sl(2)$ algebra, recall the (Gauss) decomposition formula.

9.7. LEMMA.

$$e^{sR+u\rho+t\Delta}=\exp\left(\frac{sT}{d-uT}R\right)\left(\frac{d\operatorname{sech}d}{d-uT}\right)^{\rho}\exp\left(\frac{tT}{d-uT}\Delta\right), \quad (9.21)$$

where $d^2=u^2-st$ *and* $T=\tanh d$.

Proof. See e.g., [4, 12, and 15]. ∎

Thus

9.8. THEOREM.

$$e^{tH}=\exp[t(\Lambda+P)]\exp(tx_\mu\delta_\mu) \quad (9.22)$$

with

$$\exp[t(\Lambda+P)]=\exp(t\tau P)\,\sigma^{4E/|\mathbf{z}|^2}\exp(t\tau\Lambda), \quad (9.23)$$

where $\tau=\tan(t\,|\mathbf{z}|/2)$ *and* $\sigma=\sec(t\,|\mathbf{z}|/2)$.

Proof. From Lemma 9.6, in terms of the standard $sl(2)$ algebra,

$$\Lambda=\frac{|\mathbf{z}|}{2}\Delta, \qquad P=\frac{|\mathbf{z}|}{2}R, \qquad E=\frac{|\mathbf{z}|^2}{4}\rho. \quad (9.24)$$

In Lemma 9.7, put $s\to t\,|\mathbf{z}|/2$, $u\to 0$, $t\to t\,|\mathbf{z}|/2$, and $d\to it\,|\mathbf{z}|/2$. With τ and σ as stated above, the result stated in the second part of the theorem follows. Since $H=\Lambda+P+x_\mu\delta_\mu$, one needs only to check that $[\Lambda+P, x_\mu\delta_\mu]=0$, which follows readily using the skew-symmetry of A. ∎

This gives an explicitly calculable formula for the heat flow on polynomial functions on groups of type-H. In conjunction with Theorem 9.4, it gives as well formulas for the joint moments of $w_1, ..., w_m$ and the stochastic integrals $\int_0^t w_\rho\,dw_\sigma\,K^{\rho\sigma}$. It only remains to remark on the action of e^{tH} given by Theorem 9.8 on functions of $x_1, ..., x_m$. The actions of Λ, P, and E are clear from (9.16). For $x_\mu\delta_\mu$, note that

$$\left.\frac{d}{dt}\right|_0 f(\mathbf{x}e^{tA})=x_\mu A^{\mu\lambda}D_\lambda f(\mathbf{x})=x_\mu\delta_\mu f(\mathbf{x}), \quad (9.25)$$

giving the required relation, namely

$$e^{tx_\mu\delta_\mu}f(\mathbf{x})=f(\mathbf{x}e^{tA}). \quad (9.26)$$

10. Concluding Remarks

In this paper we have presented a new technique for factorization of group elements. An original method of dual representations has been used for finding polynomial representations of the corresponding groups. By our method polynomial solutions of heat equations are explicitly computable. The key point for calculating the heat flow is reduction to $sl(2)$ in conjunction with the Feynman–Kac formula.

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