Structure of matrix manifolds and a particle model

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The decomposition of matrix manifolds into homogeneous spaces of certain groups is studied in some detail. The results are applied to the derivation of the internal structure of $SU(2,2)\times SU(m)$- and $P_4\times SU(m)$-invariant particle models where the first (second) factor in the direct product represents external (internal) symmetry. © *1996 American Institute of Physics.* [S0022-2488(96)03302-1]

I. INTRODUCTION

The mathematical description of physical laws is based on observed symmetries and underlying geometry. An example is the Poincaré symmetry $P_4=T_4\times_s SO(3,1)$, with the underlying space–time M_4, one of the homogeneous spaces of P_4, namely

$$M_4\cong T_4\cong P_4/SO(3,1). \tag{1.1}$$

This fact inspired some physicists[1,2] to investigate other homogeneous spaces of the Poincaré group

$$\frac{P_4}{H_i}\cong\frac{P_4}{SO(3,1)}\cdot\frac{SO(3,1)}{H_i},\quad H_i\subset SO(3,1),\quad i=1,2,\ldots \tag{1.2}$$

for possible applicability in physics. For instance, the local coordinates on $SO(3,1)/H_i$ can be interpreted as internal degrees of freedom of a relativistic particle. In this paper, we wish to combine an old idea[3] of describing the particle structure in complex space rather than in Minkowski space, with the investigation of the homogeneous spaces of the entire physical symmetry group, comprising both types of symmetries, internal as well as external.[4] We shall assume, in accordance with experiment, that the physical symmetry is the direct product $SU(2,2)\times SU(m)$ or $P_4\times SU(m)\subset SU(2,2)\times SU(m)$ of external conformal or Poincaré and internal $SU(m)$ symmetry (we keep m arbitrary to include such possibilities as $SU(3)$, $SU(3)\times SU(2)\times U(1)$, etc.). The Poincaré group P_4 is here considered as a subgroup of $SU(2,2)$. The natural representation space for a direct product $GL(n,\mathbf{C})\times GL(m,\mathbf{C})$ is a complex matrix manifold $\mathbf{C}^{nm}$. In the case of $SU(2,2)\times SU(m)$ it will be $\mathbf{C}^{4m}$. In this space, both internal and external symmetries have a common geometrical basis in contrast to space–time, where only external symmetries are geometrized.

We shall consider homogeneous manifolds of $SU(2,2)\times SU(m)$ and $P_4\times SU(m)$ in $\mathbf{C}^{4m}$ and show that there exists one and only one such manifold which admits a unique and consistent projection onto the compactified complex Minkowski space.

In the case of smaller $P_4\times SU(m)$ symmetry we arrive at the homogeneous manifold

$$\frac{P_4\times SU(m)}{SO(2)\times SU(m-2)}\cong\frac{P_4}{SO(3,1)}\times\frac{SO(3,1)\times SU(m)}{SO(2)\times SU(m-2)}. \tag{1.3}$$

A particle structure in this model is described by a 5-dimensional real manifold $SO(3,1)/SO(2)$ and the manifold $SU(m)/SU(m-2)$, depending on the kind of internal symmetry.

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The structure of homogeneous submanifolds of $\mathbf{C}^{nm}$ with arbitrary n and m can be investigated, up to a certain point, without essential difficulties (Sections II–IV). This general case provides the theory of n complex m-vectors subject to certain invariance conditions and generalizes, in a certain sense, the theory of spinors (including bispinors, twistors etc.) to arbitrary dimensions. When the symmetry is the direct product of more than two groups, one has to generalize to "tensor manifolds" of tensors with more than two indices.[5] One can also consider supermatrices (supertensors) being representation spaces of direct products of supergroups and their decomposition into homogeneous structures.[6]

In the case of sets of vector fields, the general theory provides a classification of all possible invariance constraints.

In Sections V and VI, we derive the internal structure of $SU(2,2)\times SU(m)$-invariant particle models determined by the above-mentioned assumptions. This structure is described in terms of homogeneous spaces (like (1.3)). It remains to describe invariant dynamics and invariant differential operators in these spaces; these questions will be treated elsewhere.

II. MATRIX MANIFOLDS

Let us consider the set $\mathbf{C}^{nm}$ of all complex $n\times m$ matrices. The elements of this set may be viewed as arrays of m complex n-vectors (or m complex n-vectors) or, more geometrically, as homomorphisms $\mathrm{Hom}(\mathbf{C}^m,\mathbf{C}^n)$ of the vector spaces, i.e., linear maps $\mathbf{C}^m\rightarrow\mathbf{C}^n$. In the context of chosen bases in $\mathbf{C}^n$ and $\mathbf{C}^m$, the elements of $\mathbf{C}^{nm}$ will be represented by matrices indexed

$$M=\{m_{a\alpha}\}^{a=1,\dots,n;}_{\alpha=1,\dots,m}\in\mathbf{C}^{nm}\cong\mathrm{Hom}(\mathbf{C}^m,\mathbf{C}^n).$$

The set $\mathbf{C}^{nm}\cong\mathrm{Hom}(\mathbf{C}^m,\mathbf{C}^n)$ decomposes in a natural way into submanifolds $\mathscr{O}_k^{(n,m)}$ of matrices (maps) of fixed rank k

$$\mathscr{O}_k^{(n,m)}:=\{M\in\mathbf{C}^{nm}:\mathrm{rank}M=k\}. \tag{2.1}$$

The decomposition is given by

$$\mathbf{C}^{nm}=\bigcup_{k=0}^{\min(n,m)}\mathscr{O}_k^{(n,m)},\quad \mathscr{O}_k^{(n,m)}\cap\mathscr{O}_l^{(n,m)}=\delta_{kl}\mathscr{O}_k^{(n,m)}. \tag{2.2}$$

A matrix of rank k is characterized by the fact that all subdeterminants (minors)

$$m\begin{pmatrix}\alpha_1 & ,\dots, & \alpha_l\\ a_1 & ,\dots, & a_l\end{pmatrix}:=\det\begin{bmatrix}m_{a_1\alpha_1} & \cdots & m_{a_1\alpha_l}\\ \vdots & & \\ \vdots & & \\ m_{a_l\alpha_1} & \cdots & m_{a_l\alpha_l}\end{bmatrix} \tag{2.3}$$

of order l higher than k vanish, and that there exists at least one nonvanishing subdeterminant of order k, say

$$m\begin{pmatrix}\alpha_1 & \cdots & \alpha_k\\ a_1 & \cdots & a_k\end{pmatrix}\neq 0. \tag{2.4}$$

Equation (2.4) determines a coordinate neighborhood on the manifold $\mathscr{O}_k^{(n,m)}$. There are $\binom{n}{k}\binom{m}{k}$ such neighborhoods according to the $\binom{n}{k}$ possibilities to choose k rows out of n rows, and the $\binom{m}{k}$ possibilities to choose k columns out of m columns.

Let us choose on $\mathscr{O}_k^{(n,m)}$ a neighborhood corresponding to a particular $k\times k$ square submatrix

$$K=\begin{bmatrix} m_{a_1\alpha_1} & \dots & m_{a_1\alpha_k} \\ \vdots & & \vdots \\ m_{a_k\alpha_1} & \dots & m_{a_k\alpha_k} \end{bmatrix}, \quad \det = K \neq 0, \tag{2.5}$$

where $a_1 < a_2 < \dots < a_k$ and $\alpha_1 < \alpha_2 < \dots < \alpha_k$ are some distinct k numbers from set $\{1,2,\dots,n\}$ and $\{1,2,\dots,m\}$, respectively. Denote the complementary ordered subsets by $a_{k+1} < \dots < a_n$ and $\alpha_{k+1} < \dots < \alpha_n$, respectively, and denote the complementary matrices by

$$A=\begin{bmatrix} m_{a_{k+1}\alpha_1} & \dots & m_{a_{k+1}\alpha_k} \\ \vdots & & \vdots \\ m_{a_n\alpha_1} & \dots & m_{a_n\alpha_k} \end{bmatrix}, \quad B=\begin{bmatrix} m_{a_1\alpha_{k+1}} & \dots & m_{a_1\alpha_m} \\ \vdots & & \vdots \\ m_{a_k\alpha_{k+1}} & \dots & m_{a_k\alpha_m} \end{bmatrix},$$

$$Y=\begin{bmatrix} m_{a_{k+1}\alpha_{k+1}} & \dots & m_{a_{k+1}\alpha_m} \\ m_{a_n\alpha_{k+1}} & \dots & m_{a_n\alpha_m} \end{bmatrix}. \tag{2.6}$$

Due to the fact that, on $\mathcal{O}_k^{(n,m)}$, $\det K \neq 0$ and all higher order subdeterminants vanish, we have, according to well known properties from linear algebra,

$$A = aK, \quad Y = aB, \quad B = Kb, \quad Y = Ab, \tag{2.7}$$

where

$$a = \{a^{a_i}_{a_j}\}_{\substack{i=1,\dots,k \\ j=k+1,\dots,n}} \qquad b = \{b^{\alpha_i}_{\alpha_j}\}_{\substack{i=1,\dots,k \\ j=k+1,\dots,n}}. \tag{2.8}$$

These formulae express the fact that, in a $n \times m$ matrix of rank k, $n-k$ rows are linear combination of the remaining k rows.

The decomposition of $M \in \mathcal{O}_k^{(n,m)}$ into K, A, B, and Y is particularly simple in the neighborhood determined by $m\binom{1,\dots,k}{1,\dots,k}$. We have in this case

$$M=\begin{bmatrix} K & B \\ A & Y \end{bmatrix}. \tag{2.9}$$

It is sufficient to consider this particular case without loss of generality; the general formulae can be obtained by simply replacing the submatrices K, A, B, and Y in (2.9) by the general submatrices (2.5), (2.6). Due to invertibility of K ($\det K \neq 0$) we obtain from (2.7)

$$Y = aKb = AK^{-1}B. \tag{2.10}$$

These formulae, as well as formulae (2.7), describe the possible natural coordinate systems on $\mathcal{O}_k^{(n,m)}$ corresponding to the neighborhood $\det K \neq 0$. Coordinates a and b play a particularly important role because of their invariance properties. Coordinates a do not depend on the particular selection of columns (Greek indices), while coordinates b do not depend on the selection of rows (Latin indices).

The $(n-k)(m-k)$ dependent coordinates Y are functions of the $k^2 + k(n-k) + k(m-k)$ independent coordinates, say K,A,B. The complex dimension of $\mathcal{O}_k^{(n,m)}$ is, therefore,

$$\dim \mathcal{O}_k^{(n,m)} = k(n+m-k). \tag{2.11}$$

$\dim O_k^{(n,m)} = k\times(n+m-k)$

$0 \leq k \leq \min(n,m)$

	n+m	k=0	1	2	3	4	5	6	7	8	9	...
	0	0										
	1	0	0									
	2	0	1	0								
	3	0	2	2	0							
	4	0	3	4	3	0						
	5	0	4	6	6	4	0					
n = 4, m = 2 →	6	0	5	8	9	8	5	0				
	7	0	6	10	12	12	10	6	0			
n = 4, m ≥ 4 →	8	0	7	12	15	16	15	12	7	0		
	9	0	8	14	18	20	20	18	14	8	0	
	10	0	9	16	21	24	25	24	21	16	9	0
	...	...										

FIG. 1. Dimensions of matrix manifolds.

The manifold of the lowest dimension, corresponding to $k=0$, is a point $m_{a\alpha}=0$ for all $a=1,\ldots,n;\alpha=1,\ldots,m$. The next admissible dimension is already $n+m-1$. The dimension $k(n+m-k)$ of $\mathscr{O}_k^{(n,m)}$ appears fairly often in our considerations, so let us draw[7] the analogue of the Pascal triangle for the quantity

$$\begin{bmatrix} a \\ b \end{bmatrix} := b(a-b) \tag{2.12}$$

(see Figure 1). Dimensions of $\mathscr{O}_k^{(n,m)}$ appear here as $[{}^{n+m}_{k}]$. In the case $n=4$, $m\geqslant 4$ we have a decomposition

$$\mathbf{C}^{4m} = \mathscr{O}_0^{(4,m)} \cup \mathscr{O}_1^{(4,m)} \cup \mathscr{O}_2^{(4,m)} \cup \mathscr{O}_3^{(4,m)} \cup \mathscr{O}_4^{(4,m)} \tag{2.13}$$

with corresponding complex dimensions $k(4+m-k)$ with $k=1,\ldots,4$. The bounded region in Figure 1 displays possible dimensions; the right hand side of the triangle is cut out by the requirement $k\leqslant\min(n,m)$. The case $n=4,m=2$ is also indicated in Figure 1 since it corresponds to the Penrose model.[8] Definition (2.1–2) implies that each manifold $\mathscr{O}_l^{(n,m)}$ with $l<k$ lies in the boundary of $\mathscr{O}_k^{(n,m)}$ in the sense

$$\mathscr{O}^l_{(n,m)} \subset \bar{\mathscr{O}}_k^{(n,m)}, \quad l<k, \tag{2.14}$$

where the "bar" denotes closure in the topology induced on $\mathscr{O}_k^{(n,m)}$ from the natural topology in $\mathbf{C}^{nm}$. We can write therefore

$$\bar{\mathscr{O}}_k^{(n,m)} = \bigcup_{l=0}^{k} \mathscr{O}_l^{(n,m)}. \tag{2.15}$$

Each manifolds $\mathscr{O}^l_{(n,m)}$ has elements arbitrarily close to $\mathscr{O}:=\mathscr{O}_0^{(n,m)}$, and together they form a flag of manifolds[7] (see Figure 2) in the sense that $\bar{\mathscr{O}}_k^{(n,m)} \subset \bar{\mathscr{O}}_{k+1}^{(n,m)}$. All closed varieties $\bar{\mathscr{O}}_k^{(n,m)}$ meet at the point $\mathscr{O}$ and their tangent spaces at this point form a flag of spaces in the usual sense

$$\mathscr{O} = T_0\bar{\mathscr{O}}_0^{(n,m)} < T_0\bar{\mathscr{O}}_1^{(n,m)} < \ldots < T_0\bar{\mathscr{O}}_k^{(n,m)} < \ldots < T_0\mathscr{O}_{\min(n,m)}^{(n,m)} \cong \mathbf{C}_{nm} \tag{2.16}$$

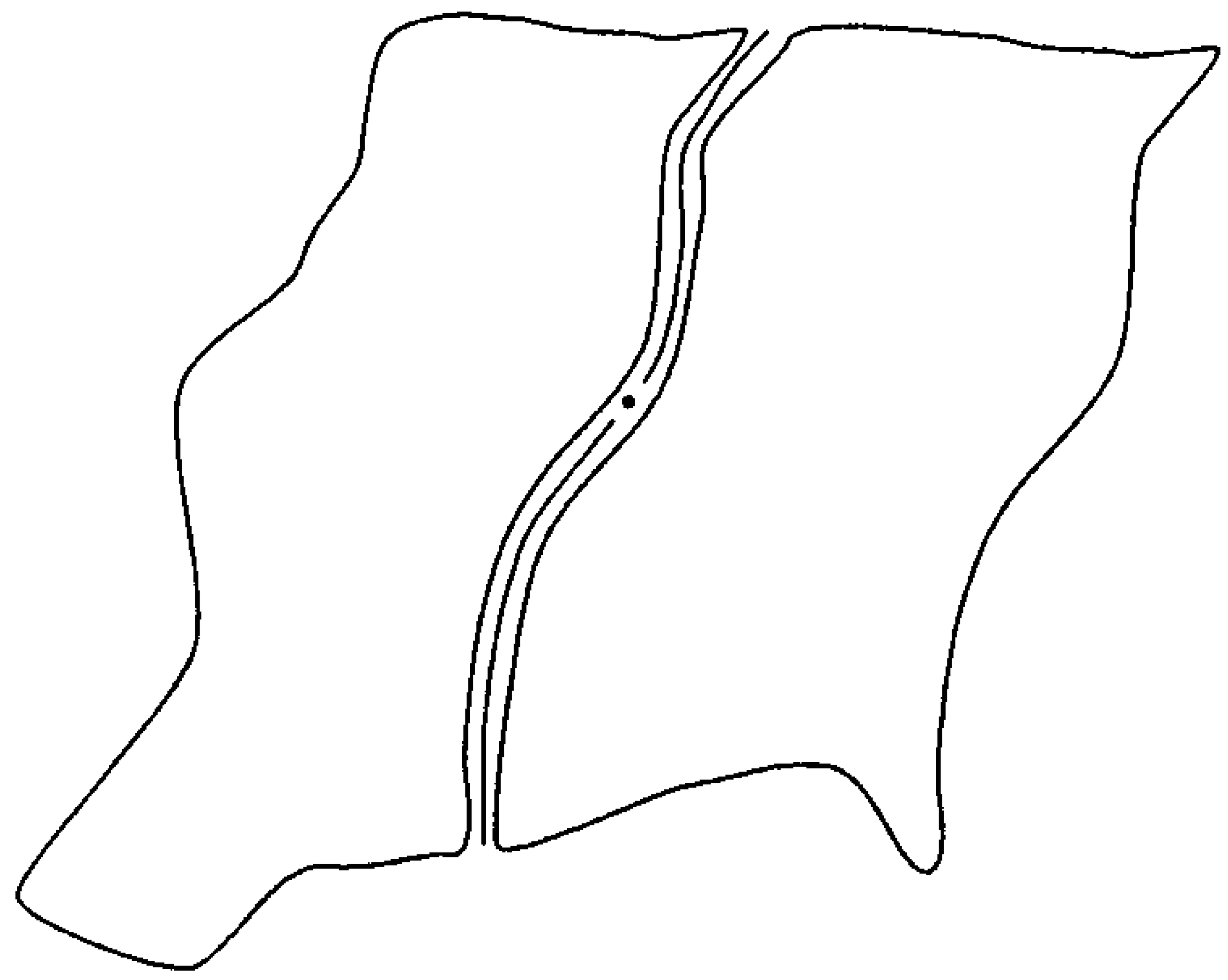

FIG. 2. Matrix manifolds form a "flag."

being an element of the flag manifold

$$F_{[{n+m \atop 0}],[{n+m \atop 1}],\ldots,[{n+m \atop \min(n,m)}]}. \tag{2.17}$$

According to the interpretation of $\mathbf{C}^{(n,m)}$ as $\mathrm{Hom}(\mathbf{C}^m,\mathbf{C}^n)$, matrix M belonging to the submanifold $\mathscr{O}_k^{(n,m)}$ represents a homomorphism $\mathbf{C}^m \to \mathbf{C}^n$; its kernel Ker M is an $(m-k)$-dimensional subspace of $\mathbf{C}^m$ and its image Im M is a k-dimensional subspace of $\mathbf{C}^n$. This is particularly clear in the example of a matrix

$$M_0 = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} \in \mathscr{O}_k^{(n,m)}. \tag{2.18}$$

Thus, the canonical decomposition of a homomorphism gives in our case[7]

$$\mathbf{C}^m \xrightarrow{\pi} \mathbf{C}^m/\mathrm{Ker}\, M \xrightarrow{\iota} \mathrm{Im}\, M \xrightarrow{\epsilon} \mathbf{C}^n \tag{2.19}$$

(cf., Figure 3) where π is a canonical projection onto the coset space, ι is an isomorphism, and ε is an embedding. This suggests that, at least locally, manifold $\mathscr{O}_k^{(n,m)}$ may be decomposed into

$$\mathscr{S}^m_{m-k} \times GL(k,\mathbf{C}) \times \mathscr{S}^n_k, \tag{2.20}$$

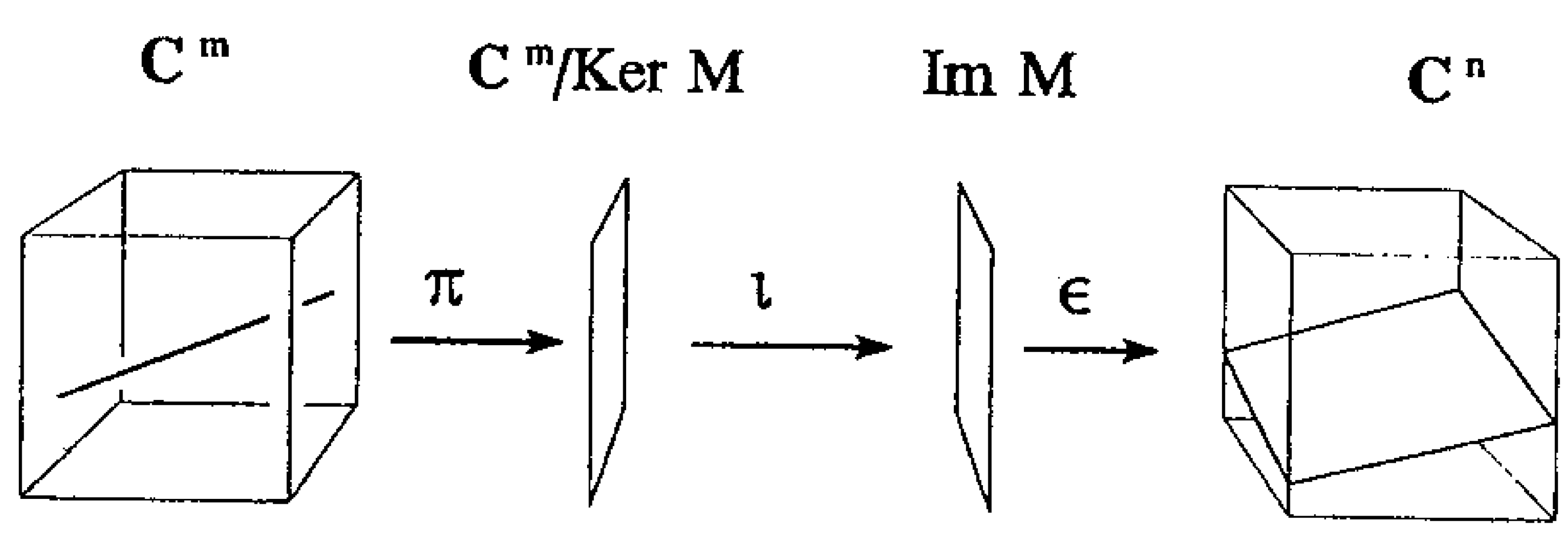

FIG. 3. Three maps.

where $\mathcal{G}^m_{m-k}$ and $\mathcal{G}^n_k$ are complex Grassman manifolds consisting of all $(m-k)$-dimensional subspaces in $\mathbf{C}^m$ and all k-dimensional subspaces in $\mathbf{C}^n$, respectively; group $\mathrm{GL}(k,\mathbf{C})$ is the k^2-dimensional set of all isomorphisms between the k-dimensional spaces $\mathbf{C}^m/\mathrm{Ker}\,M$ and $\mathrm{Im}\,M$.

However, globally manifold $\mathcal{O}_k^{(n,m)}$ cannot be trivialized into a direct product (2.20). Instead, one may consider $\mathcal{O}_k^{(n,m)}$ as a fiber bundle over $\mathcal{G}^m_{m-k}\times\mathcal{G}^n_k$ with the group $\mathrm{GL}(k,\mathbf{C})$ as the typical fiber. Including the natural projections of the Cartesian product of the two Grassman manifolds, we obtain the following diagram

$$\begin{array}{ccccc} & & \mathcal{O}_k^{(n,m)} & & \\ & \pi_1 \swarrow & \downarrow \pi & \searrow \pi_2 & \\ \mathcal{G}^m_{m-k} & \xleftarrow{(\mathrm{id},0)} & \mathcal{G}^m_{m-k}\times\mathcal{G}^n_k & \xrightarrow{(0,\mathrm{id})} & \mathcal{G}^n_k \end{array}$$

of three possible fiberings of manifold $\mathcal{O}_k^{(n,m)}$:

$$(\mathcal{O}_k^{(n,m)},\mathcal{G}^m_{m-k}\times\mathcal{G}^n_k,\pi),\quad (\mathcal{O}_k^{(n,m)},\mathcal{G}^m_{m-k},\pi_1),\quad (\mathcal{O}_k^{(n,m)},\mathcal{G}^n_k,\pi_2)$$

where the projections are defined

$$\begin{aligned} \pi&:\mathcal{O}_k^{(n,m)}\to\mathcal{G}^m_{m-k}\times\mathcal{G}^n_k : M\to\mathrm{Ker}\,M\times\mathrm{Im}\,M \\ \pi_1&:\mathcal{O}_k^{(n,m)}\to\mathcal{G}^m_{m-k} : M\to\mathrm{Ker}\,M \\ \pi_2&:\mathcal{O}_k^{(n,m)}\to\mathcal{G}^n_k : M\to\mathrm{Im}\,M \end{aligned} \tag{2.21}$$

and the typical fibers are isomorphic to $\mathrm{GL}(k,\mathbf{C})$, $\mathbf{C}^{km}$, and $\mathbf{C}^{kn}$, respectively.

In local coordinates, Grassman manifolds $\mathcal{G}^m_{m-k}$ and $\mathcal{G}^n_k$ are parameterized by elements of the matrices a and b respectively, and the fiber $\mathrm{GL}(k,\mathbf{C})$ is parameterized by elements of the matrix K (cf., (2.10)). The fiber bundle $\mathcal{O}_k^{(n,m)}$ is not trivializable. In particular, it also contains all the topological singularities of the Grassman manifold in the base.

Remark: note that since the dimension of a complex Grassman manifold is $\dim\mathcal{G}^n_k=k(n-k)$, the complex dimension of $\mathcal{O}_k^{(n,m)}$ is, according to (2.20), equal to $k(m-k)+k(n-k)+k^2=k(n+m-k)$, in a complete agreement with (2.11).

III. GROUP THEORETICAL DESCRIPTION OF MATRIX MANIFOLDS

Space $\mathbf{C}^{nm}\cong\mathrm{Hom}(\mathbf{C}^m,\mathbf{C}^n)$ is a natural representation space for the direct product $GL(n,\mathbf{C})\times GL(m,\mathbf{C})$:

$$M\to M'=gMh^{-1},\quad g\in GL(n,\mathbf{C}),\quad h\in GL(m,\mathbf{C}), \tag{3.1}$$

according to the commuting diagram

$$\begin{array}{ccc} \mathbf{C}^m & \xrightarrow{M} & \mathbf{C}^n \\ h\downarrow & & \downarrow g \\ \mathbf{C}^m & \xrightarrow{M'} & \mathbf{C}^n \end{array} \tag{3.2}$$

The manifolds $\mathscr{O}_k^{(n,m)}$ are orbits of $\mathrm{GL}(n,\mathrm{C})\times \mathrm{GL}(m,\mathrm{C})$. Indeed, a linear transformation of rows and columns does not change the rank of matrix. Moreover, the group $\mathrm{GL}(n,\mathrm{C})\times\mathrm{GL}(m,\mathrm{C})$ acts transitively on each $\mathscr{O}_k^{(n,m)}$. We can describe therefore the matrix manifolds $\mathscr{O}_k^{(n,m)}$ as homogeneous spaces[9] of the group $\mathrm{GL}(n,\mathrm{C})\times\mathrm{GL}(m,\mathrm{C})$:

$$\mathscr{O}_k^{(n,m)}=\frac{\mathrm{GL}(n,\mathrm{C})\times\mathrm{GL}(m,\mathrm{C})}{H_k^{(n,m)}}, \tag{3.3}$$

where $H_k^{(n,m)}$ is the isotropy group of the group $\mathrm{GL}(n,\mathrm{C})\times\mathrm{GL}(m,\mathrm{C})$ in $\mathscr{O}_k^{(n,m)}$. For the point

$$M_0=\begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix}\in \mathscr{O}_k^{(n,m)}$$

the isotropy group $H_k^{(n,m)}$ can be easily calculated

$$H_k^{(n,m)}=\begin{bmatrix} g_1 & g_2 \\ 0 & g_1 \end{bmatrix}\times\begin{bmatrix} g_1^{-1} & 0 \\ h_2 & h_3 \end{bmatrix}, \tag{3.4}$$

where $g_1\in \mathrm{GL}(k,\mathrm{C})$, $g_2\in \mathrm{C}^{k(n-k)}$, $g_3\in\mathrm{GL}(n-k,\mathrm{C})$, $h_2\in\mathrm{C}^{(m-k)k}$, $h_3\in\mathrm{GL}(m-k,\mathrm{C})$. We check the (complex) dimensions: $\dim \mathrm{GL}(n,\mathrm{C})\times\mathrm{GL}(m,\mathrm{C})/H_k^{(n,m)}=n^2+m^2-k^2-(n-k)^2-(m-k^2)-k(n-k)-k(m-k)=k(n+m-k)$.

Another, equivalent, group-theoretical description of $\mathscr{O}_k^{(n,m)}$ can be obtained by representing the Grassman manifolds appearing in the base of the fiber bundle $\mathscr{O}_k^{(n,m)}$ as homogeneous spaces of certain groups. There are two possibilities to represent a complex Grassman manifold as a homogeneous space:

$$\mathscr{G}_k^n\cong\frac{\mathrm{U}(n)}{\mathrm{U}(k)\mathrm{U}(n-k)}\cong\frac{\mathrm{GL}(n,\mathrm{C})}{H_k^n} \tag{3.5}$$

where H_k^n is a matrix group defined:

$$H_k^n=\left\{\begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix},\quad g_1\in\mathrm{GL}(k,\mathrm{C}),\quad g_2\in\mathrm{C}^{k(n-k)},\quad g_3\in\mathrm{GL}(n-k,\mathrm{C})\right\}. \tag{3.6}$$

It is important to notice that Grassman manifolds $\mathscr{G}_k^n$ and $\mathscr{G}_{m-k}^m$ of the base of the fiber bundle $\mathscr{O}_k^{(n,m)}$ are invariant under action of the groups $1_n\times\mathrm{GL}(m,\mathrm{C})$ and $\mathrm{GL}(n,\mathrm{C})\times 1_m$, respectively, and transform into themselves under action of the groups $\mathrm{GL}(n,\mathrm{C})\times 1_m$ and $1_n\times\mathrm{GL}(m,\mathrm{C})$, respectively. This follows immediately from the remark after formula (2.10) and (2.21) stating that the coefficients a (b) of linear combinations $A=aK$ $(B=Kb)$ do not depend on the columns (rows) of matrix M.

Let us consider, finally, the transformation properties of the local coordinates on $\mathscr{O}_k^{(n,m)}$ with respect to the group $\mathrm{GL}(n,\mathrm{C})\times\mathrm{GL}(m,\mathrm{C})$. For this purpose, to simplify the notation, we extend the $(n-k)\times k$ and $k\times(m-k)$ matrices a and b of (2.8) to the $n\times k$ and $k\times m$ matrices

$$a=\{a^{a_i}_{a_j}\}^{i=1,\dots,k}_{j=1,\dots,n},\quad b=\{b^{a_i}_{\alpha_j}\}^{i=1,\dots,k}_{j=1,\dots,n} \tag{3.7}$$

by means of unit matrices

$$\{a^{a_i}_{a_j}\}=\{\delta^{a_i}_{a_j}\},\quad \{b^{\alpha_i}_{\alpha_j}\}=\{\delta^{\alpha_i}_{\alpha_j}\}\quad i,j=1,\dots,k. \tag{3.8}$$

(We denote these extended matrices by the same letters; the context will always clarify whether we have to do with the original or the extended matrices.) With the help of the extended matrices we can combine relations (2.7), (2.11) into a single formula

$$M = aKb, \quad M \in \mathscr{O}_k^{(n,m)}, \quad \det K \neq \mathscr{O}. \tag{3.9}$$

Equation (3.1) takes, on $\mathscr{O}_k^{(n,m)}$, the form

$$M' = gaKbh^{-1} = a'K'b', \tag{3.10}$$

where K' is chosen so that det $K \neq 0$ (this is always possible because transformation (3.1) does not change the rank of M). Matrices a' and b' have, in this neighborhood, the same functional dependence of the elements of M' as the matrices a and b have of the elements of M:

$$\begin{aligned} a' &= a(gMh^{-1}) = a(gM), \\ b' &= b(gMh^{-1}) = b(Mh^{-1}). \end{aligned} \tag{3.11}$$

The second parts of these equalities express the invariance of a and b with respect to $1_n \times \mathrm{GL}(m,\mathrm{C})$ and $\mathrm{GL}(n,\mathrm{C}) \times 1_m$, respectively. (Cf., the remark after formula (2.6)). It is also useful to note the explicit form of the transformation law (3.11). For this purpose, we write equation $a = AK^{-1}$ as

$$a^{a_i}_{a_j} = \sum_{l=1}^{k} m_{a_j \alpha_l} (K^{-1})^{\alpha_l a_i} = \frac{m\begin{pmatrix} \alpha_1 \ldots & & \ldots \alpha_k \\ a_1 \ldots & a_{i-1} a_j a_{i+1} & \ldots a_k \end{pmatrix}}{m\begin{pmatrix} \alpha_1 & \ldots & \alpha_k \\ a_1 & \ldots & a_k \end{pmatrix}}, \tag{3.12}$$

where $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. The same formula holds for a' in terms of M'

$$\begin{aligned} a'^{ai}_{aj} &= \frac{m'\begin{pmatrix} \alpha_1 \ldots & & \ldots \alpha_k \\ a_1 \ldots & a_{i-1} a_j a_{i+1} & \ldots a_k \end{pmatrix}}{m'\begin{pmatrix} \alpha_1 & \ldots & \alpha_k \\ a_1 & \ldots & a_k \end{pmatrix}} = \frac{gaKbh^{-1}\begin{pmatrix} \alpha_1 \ldots & & \ldots \alpha_k \\ a_1 \ldots & a_{i-1} a_j a_{i+1} & \ldots a_k \end{pmatrix}}{gaKbh-1\begin{pmatrix} \alpha_1 & \ldots & \alpha_k \\ a_1 & \ldots & a_k \end{pmatrix}}, \\ &= \frac{ga\begin{pmatrix} \alpha_1 \ldots & & \ldots \alpha_k \\ a_1 \ldots & a_{i-1} a_j a_{i+1} & \ldots a_k \end{pmatrix}}{ga\begin{pmatrix} \alpha_1 & \ldots & \alpha_k \\ a_1 & \ldots & a_k \end{pmatrix}} \end{aligned} \tag{3.13}$$

where $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. Analogously

$$b^{\alpha_i}_{\alpha_j} = \frac{m\begin{pmatrix} \alpha_1 \ldots & \alpha_{i-1} \alpha_j \alpha_{i+1} & \ldots \alpha_k \\ \alpha_1 \ldots & & \ldots \alpha_k \end{pmatrix}}{m\begin{pmatrix} \alpha_1 & \ldots & \alpha_k \\ a_1 & \ldots & a_k \end{pmatrix}} \tag{3.14}$$

and

$$b'^{\alpha_i}_{\alpha_j} = \frac{bh^{-1}\begin{pmatrix} \alpha_1 \ldots & \alpha_{i-1}\alpha_j\alpha_{i+1} & \ldots \alpha_k \\ a_1 \ldots & & \ldots a_k \end{pmatrix}}{bh^{-1}\begin{pmatrix} \alpha_1 \ldots & \ldots & \alpha_k \\ a_1 & \ldots & a_k \end{pmatrix}}. \tag{3.15}$$

Relations (3.13) and (3.15) provide an alternative proof of the transformation properties of a and b and an explicit form of the transformation law. The second equation of (3.11) follows from the well known formula for subdeterminants of a product of matrices two A and B

$$AB\begin{pmatrix} j_1 & \ldots & j_l \\ i_1 & \ldots & i_l \end{pmatrix} = \sum_{\binom{n}{l}} A\begin{pmatrix} k_1 & \ldots & k_l \\ i_1 & \ldots & i_l \end{pmatrix} B\begin{pmatrix} j_1 & \ldots & j_l \\ k_1 & \ldots & k_l \end{pmatrix} \tag{3.16}$$

that is taken for the particular case $l=n$. Here the sum goes over all $\binom{n}{l}$ selections of l distinct numbers $k_1,\ldots,k_l$ between 1 and n.

Let us consider the infinitesimal form or the transformation law of the local coordinates. The infinitesimal version of (3.10) and (3.11) is

$$M' = M + \delta M, \quad a' = a + \delta a, \quad b' = b + \delta b, \tag{3.17}$$

with

$$\delta M = \delta g M + M\,\delta h^{-1} = i\sum_{k=1}^{r} \delta\lambda_k x_k M - i\sum_{k=1}^{s} \delta\mu_k M y_k \tag{3.18}$$

and

$$\delta a = \sum_{k=1}^{r} \delta\lambda_k \frac{\partial a(gM)}{\partial\lambda_k}\bigg|_{\lambda=0}, \quad \delta b = \sum_{k=1}^{s} \delta\mu_k \frac{\partial b(Mh^{-1})}{\partial\mu_k}\bigg|_{\mu=0} \tag{3.19}$$

where

$$x_k = \frac{1}{i}\frac{\partial g}{\partial\lambda_k}\bigg|_{\lambda=0}, \quad y_k = \frac{1}{i}\frac{\partial h}{\partial\mu_k}\bigg|_{\mu=0} \tag{3.20}$$

are generators of the factors GL(n,**C**) and GL(m,**C**) of the direct product GL(n,**C**) ×GL(m,**C**) or of one of its subgroups, and $\lambda_1,\lambda_2,\ldots,\lambda_r$ and $\mu_1,\mu_2,\ldots,\mu_s$ are the corresponding parameters. Formula (3.18) for δM has already its final form; the formulae for δa and δb have yet to be expressed in terms of generators x_k and y_k. Differentiate equations $A' = a'K'$ and $B' = K'b'$ with respect to the corresponding parameters:

$$\begin{aligned} \frac{\partial A'}{\partial\lambda_k} &= \frac{\partial a'}{\partial\lambda_k}K' + a'\frac{\partial K'}{\partial\lambda_k}, \\ \frac{\partial B'}{\partial\mu_k} &= \frac{\partial K'}{\partial\mu_k}b' + K'\frac{\partial b'}{\partial\mu_k}. \end{aligned} \tag{3.21}$$

This gives

$$\left.\frac{\partial a'}{\partial \lambda_k}\right|_{\lambda=0} = \left(\frac{\partial A'}{\partial \lambda_k} - a'\frac{\partial K'}{\partial \lambda_k}\right)_{\lambda=0} K^{-1} = i(x_k a - a x_k a),$$
$$\left.\frac{\partial b'}{\partial \mu_k}\right|_{\mu=0} = K^{-1}\left(\frac{\partial B'}{\partial \mu_k} - \frac{\partial K'}{\mu \lambda_k} b'\right)_{\mu=0} = -i(by_k - by_k b). \tag{3.22}$$

Substituting (3.22) into (3.19) we obtain finally

$$\delta a = i\sum_{k=1}^{r} \delta\lambda_k(x_k a - a x_k a),$$
$$\delta b = -i\sum_{k=1}^{s} \delta\mu_k(b y_k - b y_k b). \tag{3.23}$$

Similarly, one can easily show that

$$\left.\frac{\partial a'}{\partial \mu_k}\right|_{\mu=0} = \left.\frac{\partial b'}{\partial \lambda_k}\right|_{\lambda=0} = 0, \tag{3.24}$$

which confirms the transformation properties of a and b as exhibited in equation (3.11), and used explicitly in (3.19).

IV. REDUCTION OF SYMMETRY

Reduction of the symmetry $\mathrm{GL}(n,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})$ to a subgroup induces in general a decomposition of the matrix manifolds $\mathscr{O}_k^{(n,m)}$. Here, we shall consider only the case of the subgroup $\mathrm{SU}(n-p,p) \times \mathrm{SU}(m-q,q)$ determined by Hermitian metric tensors F_1 and F_2 (not necessarily diagonal), with $n-p$ and $m-q$ eigenvalues $+1$ respectively, and p and q eigenvalues -1 respectively. For simplicity of the following considerations, we shall assume that F_1 denotes Hermitian structure in the space $\mathbf{C}^n$ (two lower indices), while F_2 denotes Hermitian structure in the *dual* space $\mathbf{C}^{m*}$ (two upper indices).

Define the following invariants

$$I_n = \mathrm{Tr}\, r^n, \quad n = 1,2,\ldots, \tag{4.1}$$

where

$$r = F_2 M^* F_1 M \tag{4.2}$$

is an automorphism of $\mathbf{C}^m$ described by a non-commuting diagram

$$\begin{array}{ccc} \mathbf{C}^{m*} & \overset{M^*}{\leftarrow} & \mathbf{C}^{n*} \\ F_2\downarrow & & \uparrow F_1 \\ \mathbf{C}^m & \overset{M}{\rightarrow} & \mathbf{C}^m \end{array} \tag{4.3}$$

and where the Hermitian metric tensors F_1 and F_2 are considered as maps to the corresponding dual spaces. Clearly, the order of matrices in r can be changed by a cyclic permutation without causing change of the invariants.

One can easily show that the quantities I_n are indeed invariants of the group $SU(n-p,p)\times SU(n-q,q)$ and that they are real. Indeed, due to $g^*F_1g=F_1$ and $hF_2h^*=F_2$, we have

$$I'_n:=\mathrm{Tr}(r')^n=\mathrm{Tr}(F_2M'^*F_1M')^n=\mathrm{Tr}(F_2h^{-1*}M^*g^*F_1gMh^{-1})^n=\mathrm{Tr}(F_2M^*F_1M)^n=I_n$$

and

$$I_n^*=\mathrm{Tr}(M^*F_1^*MF_2^*)^n=\mathrm{Tr}(F_2M^*F_1M)^n=I_n.$$

Another kind of invariant appears as coefficients J of the eigenvalue equation for the automorphism r

$$\det(r-\lambda)=\sum_{n=0}^{m}(-\lambda)^n J_{m-n}=0, \tag{4.4}$$

where J's are defined

$$J_{m-n}=\frac{(-1)^n}{n!}\left.\frac{d^n\det(r-\lambda)}{d\lambda^n}\right|_{\lambda=0}. \tag{4.5}$$

These invariants have the explicit form

$$J_n=\sum_{\binom{m}{n}} r\begin{pmatrix}\alpha_1,\alpha_2,\ldots,\alpha_n\\ \alpha_1,\alpha_2,\ldots,\alpha_n\end{pmatrix}, \tag{4.6}$$

where the summation extends over all $\binom{m}{n}$ possible selections of n distinct numbers α_i from the set $\{1,2,\ldots,m\}$.

There exists a relation between both types of invariants. In case $n=1$ the invariants coincide

$$J_1=I_1 \tag{4.7}$$

and for $n\geqslant 2$ the relation is

$$J_n=\frac{1}{n!}\sum_{\alpha_1}\sum_{\alpha_2}\ldots\sum_{\alpha_n} r\begin{pmatrix}\alpha_1,\alpha_2,\ldots,\alpha_n\\ \alpha_1,\alpha_2,\ldots,\alpha_n\end{pmatrix}=\frac{1}{n!}(-1)^{n-1}(n-1)I_n+\Phi_n(I_1,I_2,\ldots,I_{n-1}), \tag{4.8}$$

with some polynomial Φ_n of order less than n. Thus, only the first k invariants are independent on $\mathcal{O}_k^{(n,m)}$. Indeed, all determinants of order higher than k vanish on $\mathcal{O}_k^{(n,m)}$ and, therefore,

$$J_{k+r}=0, \quad r=1,2,\ldots. \tag{4.9}$$

Consequently, relation

$$\frac{1}{(k+r)!}(-1)^{k+r-1}(k+r-1)I_{k+r}+\Phi_{k+r}(I_1,I_2,\ldots,I_{k+r-1})=0 \quad r=1,2,\ldots \tag{4.10}$$

provides a set of equations expressing the higher order invariants I_{k+r}, $r=1,2,\ldots$, in terms of I_1, $I_2,\ldots,I_k$. In particular

$$J_1=I_1, \quad 2J_2=I_1^2-I_2, \quad 6J_3=I_1^3-3I_1I_2+2I_3. \tag{4.11}$$

The invariants may be used to foliate the space; equations

$$I_n=\kappa_n, \quad n=1,2,\ldots,k \tag{4.12}$$

with constants κ_n satisfying certain compatibility conditions, determine a k-parametric family of $SU(n-p,p)\times SU(m-q,q)$-invariant submanifolds of space $\mathbf{C}^{nm}$. Their intersections with the $GL(n,C)\times GL(m,C)$-invariant submanifolds $\mathscr{O}_k^{(n,m)}$ provide a decomposition of $\mathscr{O}_k^{(n,m)}$ into a k-parametric family of $SU(n-p)\times SU(m-q)$-invariant submanifolds $\mathscr{O}_{(k,\kappa)}^{(n,m)}$ where $\kappa=\{\kappa_1,\kappa_2,\ldots,\kappa_k\}$. The analytic description of these sections is obtained by substituting (3.9) into (4.12):

$$I_l=\mathrm{Tr}(F_2b^*K^*a^*F_1aKb)^l=\mathrm{Tr}(f_2K^*f_1K)^l=\kappa_l^*, \quad l=1,2,\ldots,k, \tag{4.13}$$

where

$$f_1=a^*F_1a, \quad f_2=bF_2b^* \tag{4.14}$$

represent the metrics induced on the columns and rows of the matrix K from the metric tensors F_1 and F_2 on M.

The induced metric tensors f_1 and f_2 depend on the Grassman coordinates a and b respectively, on the choice of neighborhood, and on the choice of the representation of the initial metrics F_1 and F_2 on M. For the purpose of a general discussion we consider the induced metric $f(a)$ given by a $k\times k$ Hermitian matrix being a function of the $k(n-k)$ complex Grassman coordinates a. The signature of the induced metric is determined by the roots of the secular equation for f,

$$\det(f-\lambda 1)=(-\lambda)^k+(-\lambda)^{k-1}\mathrm{Tr}\, f+\ldots+\det f=\sum_{i=0}^{k}(-\lambda)^i c_{k-i}=\prod_{i=1}^{k}(\lambda_i-\lambda)=0, \tag{4.15}$$

where, similarly as in (4.4)–(4.6),

$$c_i=\sum_{\binom{k}{i}} f\begin{pmatrix} a_1,a_2,\ldots,a_n \\ a_1,a_2,\ldots,a_n \end{pmatrix}, \quad i=1,2,\ldots,k, \tag{4.16}$$

where the sum runs over all $\binom{k}{i}$ possible selections of i distinct numbers from $\{1,2,\ldots,k\}$. Roots $\lambda_i(a)$, which determine the induced metric, are functions of the Grassman coordinates a and change as the a's vary over the Grassman manifold $\mathscr{G}_k^n$.

Not all eigenvalues can appear in the induced metric. The induced metric tensor on the k-plane in C^n may have different degree of degeneration. A triple (a,b,c) will denote signature of a pluses, b minuses, and c zeros of a $(a+b+c)$-dimensional space. Notation (a,b) means that $c=0$.

Consider first the planes with non-degenerate induced metric tensors. The number of positive (negative) roots cannot exceed the number of positive (negative) signs in the original metric. If the original signature in $\mathbf{C}^n$ is $(n-p,p)$ then the admissible signatures on k-dimensional planes in $\mathbf{C}^n$ are $(k-l,l)$, for some l satisfying the obvious relations

$$k-l\leqslant n-p, \quad l\leqslant p, \quad 0\leqslant l\leqslant k \tag{4.17}$$

or, jointly,

$$L_{\min}\leqslant l\leqslant L_{\max}, \tag{4.18}$$

where

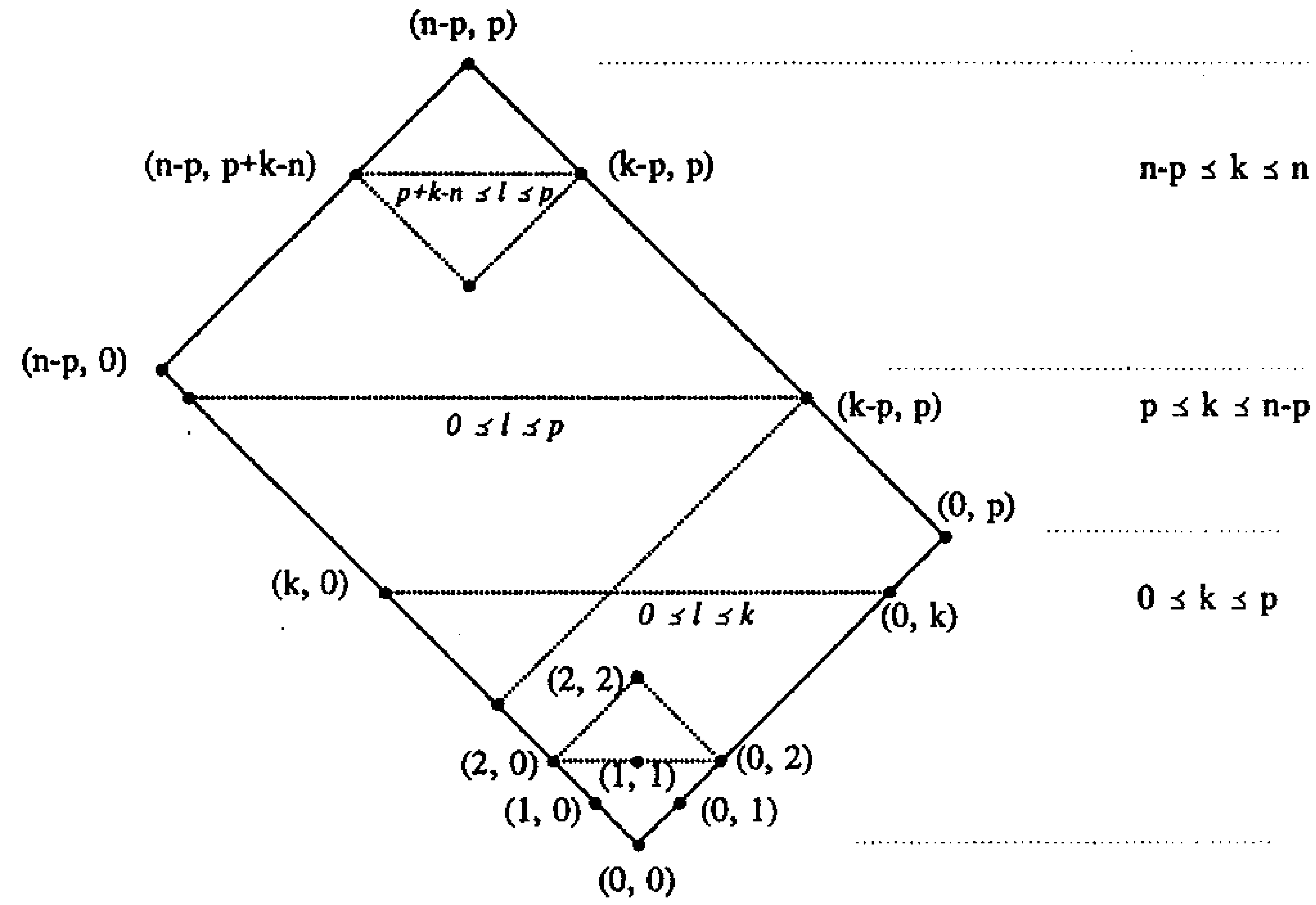

FIG. 4. The lattice of induced metric structures.

$$L_{\min}=\max\{0,k+p-n\}, \quad L_{\max}=\min\{p,k\}.$$

There are six different situations:

$$n-p\geqslant p \quad \text{and} \begin{cases} 0\leqslant k\leqslant p \\ p\leqslant k\leqslant n-p, \\ n-p\leqslant k\leqslant n \end{cases} \quad \text{or } n-p\leqslant p \quad \text{and} \begin{cases} 0\leqslant k\leqslant n-p \\ n-p\leqslant k\leqslant p \\ p\leqslant k\leqslant n. \end{cases}$$

The three possibilities corresponding to $n-p\geqslant p$ are illustrated in Figure 4.

Also, degenerate metrics will appear for certain submanifolds of $\mathscr{S}_k^n$ corresponding to zero roots $\lambda_i(a)=0$ of equation (4.13). There will be zero roots of various orders ranging from 1 to k described by the vanishing of the first coefficients $c_i(a)$ in (4.15). Root $\lambda(a)$ of multiplicity r is described by r equations

$$c_{k-j}(a)=0, \quad j=0,1,\dots,r-1. \tag{4.19}$$

Let us consider a zero root with multiplicity one. It always lies on the border between two roots of the opposite sign. The corresponding metric has signature $(k-l-1,l,1)$ and lies in $\mathscr{S}_k^n$ between two metrics differing by interchanging a positive with a negative sign, that is to say between $(k-l,l,0)$ and $(k-l-1,l+1,0)$ and may be obtain by replacing one plus of the first, or one minus of the second, by a zero.

The case of zero with multiplicity two is obtained by descending one step down to the $(k-2)$-dimensional hyperplane in $\mathbf{C}^n$ with the metric $(k-2-l,l,2)$, where now l varies $L_{\min}\leqslant l\leqslant L_{\max}-2$. Proceeding in this way we obtain, after r steps, a set of degenerate metrics of signatures

$$(k-r-l,l,r),$$

where l and r vary according to

$$L_{\min}\leqslant l\leqslant L_{\max-r}, \quad 0\leqslant r\leqslant L_{\max}-L_{\min}. \tag{4.20}$$

Summarizing, the Grassman manifold $\mathcal{G}_k^n$ decomposes into domains, each domain consisting of hyperplanes that have the same signature of the induced metric. There are $L_{\min}-L_{\max}+1$ such domains corresponding to non-degenerate metrics. They are separated by boundaries, each boundary consisting of those hyperplanes that have the same signature of the induced degenerate metric. The boundaries are determined by equations

$$c_{k-j}(a)=0, \quad j=0,1,\ldots,r-1, \quad r=1,2,\ldots,L_{\max}-L_{\min}.$$

The dimension of the boundaries decreases with r. These domains and their boundaries represent again a flag of manifolds of different dimensions meeting at the manifold with lowest dimension corresponding to $r_{\max}=L_{\min}-L_{\max}$. The latter consists of k-dimensional planes in $\mathbf{C}^n$ with induced degenerate metric of signature $(k-L_{\max},L_{\min},L_{\max}-L_{\min})$ (cf., (4.20)).

The situation is illustrated by Figure 4 (see also Ref. 7). The triangle represents a lattice $\mathcal{L}$ of metrics (i,j), the first (second) letter in the bracket denoting the number of positive (negative) signs. Lattice $\mathcal{L}$ becomes partially ordered set if we introduce the ordering relation

$$(i,j)\leqslant(i',j') \;\Leftrightarrow\; i\leqslant j' \text{ and } j\leqslant j'. \tag{4.21}$$

Let us assume that the original metric F_1 is $(n-p,p)$. The sublattice $\mathcal{L}(n-p,p)$ of all admissible induced metrics consists, according to (4.17), (4.18), of all those metrics which are in relation $\leqslant$ with the original metric $(n-p,p)$

$$\mathcal{L}(n-p,p):=\{(i,j)\in\mathcal{L}|(i,j)\leqslant(n-p,p)\}. \tag{4.22}$$

Let us pick up all the admissible metrics $\mathcal{R}^k$ which lie in the k-th row (counting from below), i.e., the row corresponding to $i+j=k$ and representing the set of non-degenerate admissible metrics induced on the k-dimensional planes from the original metric in $\mathbf{C}^n$

$$\mathcal{R}^k(n-p,p):=\{(i,j)\in\mathcal{L}(n-p,p)|i+j=k\}. \tag{4.23}$$

According to (4.20), all admissible metrics (degenerate and non-degenerate) lie in the triangle

$$\mathcal{L}^k:=\{(i,j)\in\mathcal{L}(n-p,p)|(i,j)\leqslant(i',j'),i'+j'=k\Rightarrow(i',j')\in\mathcal{R}^k\}. \tag{4.24}$$

They have to be completed up to k signs by an appropriate number of zeros. The number of metrics in the triangle $L^k(n-p,p)$ is

$$N_k^{(n-p,p)}=1+2+\ldots+(L_{\max}-L_{\min}+1)=\frac{1}{2}(L_{\max}-L_{\min}+1)(L_{\max}-L_{\min}+2). \tag{4.25}$$

Accordingly, Grassman manifold $\mathcal{G}_k^n$ decomposes into $N_k^{(n-p,p)}$ regions $\mathcal{G}_{k-r,l}^{(n-p,p)}$ of different signatures of induced metric tensors and of different dimensions

$$\mathcal{G}_k^n=\bigcup_r\bigcup_l \mathcal{G}_{k-r,l}^{(n-p,p)}, \tag{4.26}$$

where the l-summation is over the intervals indicated in (4.20). Manifold $\mathcal{O}_k^{(n,m)}$ decomposes into "chimneys" given by the inverse π^{-1} of the canonical projection π of (2.19) (cf., Figure 5)

$$\pi^{-1}(\mathcal{G}_{k-s,l_s}^{(m-q,q)}\times\mathcal{G}_{k-r,l_r}^{(n-p,p)}). \tag{4.27}$$

Introducing the invariance conditions $I_l=\kappa_l$, $l=1,\ldots,k$, we obtain finally a decomposition of the $\mathrm{SU}(n-p,p)\times\mathrm{SU}(m-q,q)$ invariant submanifolds $\mathcal{O}_k^{(n,m)}$ into domains corresponding to the various induced metrics.

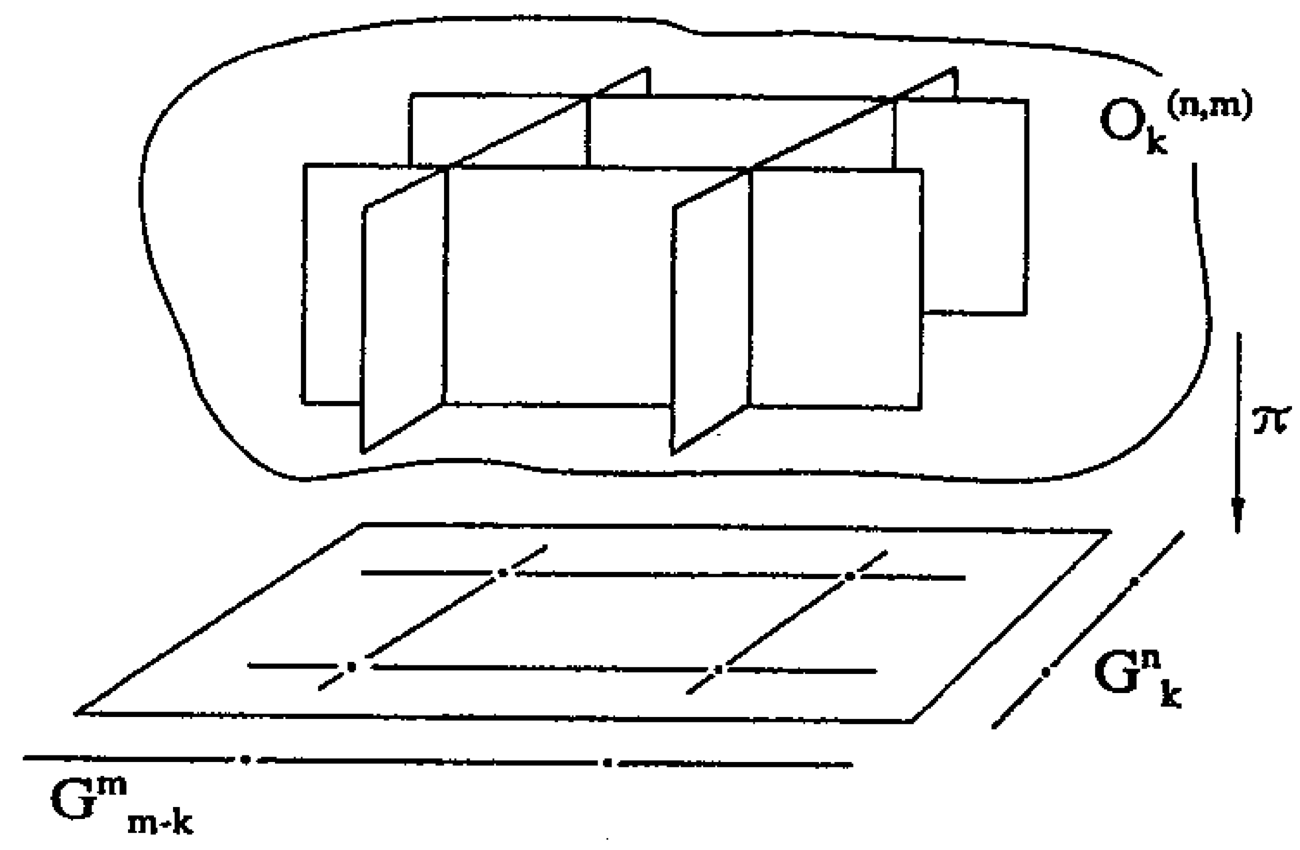

FIG. 5. Decomposition of a matrix manifold.

One may note that although the original metrics F_1 and F_2 are invariant under $SU(n-p,p)\times SU(m-q,q)$, the induced metrics f_1 and f_2 are not. This is due to the fact that the decomposition of the Grassman manifolds into subdomains depends upon the frame of reference.

V. THE MODEL

To construct a particle model one has to derive the structure of the space of internal parameters. The derivation is based on two plausible assumptions.

(1) The physical symmetry group is represented by the direct product $SU(2,2)\times SU(m)$ or its subgroup $P_4\times SU(m)$. Group SU(2,2), the covering group of the conformal group, or its Poincaré subgroup P_4, represents the external symmetries, while SU(m) represents the internal symmetries, in accordance with experimental evidence. Representing the external symmetries by SU(2,2) (or one of its subgroups) provides a common geometrical basis $\mathbf{C}^{4m}$ for both internal and external symmetries in accordance with the idea to describe physical laws in complex vector spaces [3]. So far, it is not necessary to specify m; one can think, e.g., of SU(3) or $SU(3)\times SU(2)\times U(1)$.

(2) The internal and external parameters of the particle are represented by local coordinates of an invariant homogeneous submanifold of the linear representation space $\mathbf{C}^{4m}$ of $SU(2,2)\times SU(m)$. This manifold has to satisfy the "correspondence principle:" it must admit a projection on the Minkowski space that is unique and consistent with the symmetry. We shall prove that there exists one and only one such a submanifold of $\mathbf{C}^{4m}$.

To find the manifold satisfying the above conditions we use decomposition (2.2), (2.13). For $n=4$, we have

$$\mathbf{C}^{4m}=\mathscr{O}_0^{(4,m)}\cup\mathscr{O}_1^{(4,m)}\cup\mathscr{O}_2^{(4,m)}\cup\mathscr{O}_3^{(4,m)}\cup\mathscr{O}_4^{(4,m)}$$

complemented with fiberings (2.20), (2.21). It follows immediately that the only submanifold containing $\mathscr{G}_2^4$ in the base is $\mathscr{O}_2^{(4,m)}$ with the local trivialization $\mathscr{G}_2^4\times GL(2,\mathbf{C})\times\mathscr{G}_2^m$. It is well known that $\mathscr{G}_2^4$ is homeomorphic with the compactified complex Minkowski space $\mathscr{M}_4^{\mathbf{C}}$, the homeomorphism (and its inverse) being given by the relations

$$z_\mu=\frac{i\lambda}{2}\mathrm{Tr}\sigma_\mu a,\quad a=\frac{1}{i\lambda}z_\mu\tilde{\sigma}^\mu, \tag{5.1}$$

where a is a 2×2 complex matrix the entries of which are Grassman coordinates of two-dimensional hyperplanes in $\mathbf{C}^4$. According to (2.7), we have $A=aK$ and $Y=aB$. The explicit form of the Grassman coordinates is given by equations (3.12), (3.14) specified to the particular case $k=2$, $n=4$

$$a_j^1=\frac{m\begin{pmatrix}\alpha_1 & \alpha_2\\ j & 2\end{pmatrix}}{m\begin{pmatrix}\alpha_1 & \alpha 2\\ 1 & 2\end{pmatrix}},\quad a_j^2=\frac{m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & j\end{pmatrix}}{m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & 2\end{pmatrix}},\quad j=3,4, \tag{5.2}$$

where $\sigma_0=1_2$ and σ_i, $i=1,2,3$, are Pauli matrices, $\tilde{\sigma}_i=\sigma_i$, $\tilde{\sigma}_0=-\sigma_0$. Parameter λ is a "dimensional parameter" with dimension of length and has to be introduced to relate the complex vector z_μ with the dimensionless ratios (5.2).

Following the remark after formula (2.10), coordinates a_j^k do not depend on the particular choice of (distinct) α_1 and α_2 from $\{1,2,\ldots,m\}$. This proves the uniqueness of the projection π_1.

To prove consistency with the group SU(2,2) (the a's do not transform with the respect to SU(m)), we have to investigate the transformation properties of the skew-symmetric forms appearing in (5.2)

$$m\begin{pmatrix}\alpha_1\alpha_2\\ a_1 a_2\end{pmatrix}=m_{a_1\alpha_1}m_{a_2\alpha_2}-m_{a_1\alpha_2}m_{a_2\alpha_1}=\xi_{a_1}\eta_{a_2}-\xi_{a_2}\eta_{a_1}, \tag{5.3}$$

where $a_1,a_2=1,2,3,4$, and $\alpha_1,\alpha_2=1,2,\ldots,m$. Due to the fact that the ratios of the determinants appearing in (5.2) do not depend on particular selection of (α_1,α_2), we can drop for simplicity the second index, introducing temporarily the following notation

$$\xi_a=m_{a\alpha},\quad \eta_a=m_{a\beta},\quad a=1,2,3,4. \tag{5.4}$$

In the case of $n=4$, there are six skew-symmetric forms and they satisfy one obvious relation

$$\begin{aligned}0=\frac{1}{2}m\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_1 & \alpha_2\\ a_1 & a_2 & a_3 & a_4\end{pmatrix}&=\frac{1}{8}m\begin{pmatrix}\alpha_1 & \alpha_2\\ a_1 & a_2\end{pmatrix}\varepsilon_{a_1a_2a_3a_4}m\begin{pmatrix}\alpha_1 & \alpha_2\\ a_3 & a_4\end{pmatrix}\\ &=m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & 2\end{pmatrix}m\begin{pmatrix}\alpha_1 & \alpha_2\\ 3 & 4\end{pmatrix}-m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & 3\end{pmatrix}m\begin{pmatrix}\alpha_1 & \alpha_2\\ 2 & 4\end{pmatrix}\\ &=m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & 2\end{pmatrix}m\begin{pmatrix}\alpha_1 & \alpha_2\\ 3 & 4\end{pmatrix}+m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & 4\end{pmatrix}m\begin{pmatrix}\alpha_1 & \alpha_2\\ 2 & 3\end{pmatrix}.\end{aligned} \tag{5.5}$$

It is convenient to express these forms and the coordinates κ_μ in terms of Dirac γ-matrices. In the representation

$$\gamma_\mu=i\begin{bmatrix}0 & \sigma_\mu\\ -\tilde{\sigma}_\mu & 0\end{bmatrix},\quad \varepsilon=\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}\quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,\quad \gamma_4=i\gamma_0, \tag{5.6}$$

we have the following forms

$$
\begin{aligned}
s &:= (\xi\varepsilon\eta) = m\begin{pmatrix}\alpha_1\alpha_2\\ 1\ 2\end{pmatrix} + m\begin{pmatrix}\alpha_1\alpha_2\\ 3\ 4\end{pmatrix},\\
p &:= (\xi\varepsilon\gamma_5\eta) = m\begin{pmatrix}\alpha_1\alpha_2\\ 1\ 2\end{pmatrix} - m\begin{pmatrix}\alpha_1\alpha_2\\ 3\ 4\end{pmatrix},\\
v_1 &:= (\xi\varepsilon\gamma_1\eta) = im\begin{pmatrix}\alpha_1\alpha_2\\ 1\ 3\end{pmatrix} - im\begin{pmatrix}\alpha_1\alpha_2\\ 2\ 4\end{pmatrix},\\
v_2 &:= (\xi\varepsilon\gamma_2\eta) = -m\begin{pmatrix}\alpha_1\alpha_2\\ 1\ 3\end{pmatrix} - m\begin{pmatrix}\alpha_1\alpha_2\\ 2\ 4\end{pmatrix},\\
v_3 &:= (\xi\varepsilon\gamma_3\eta) = -im\begin{pmatrix}\alpha_1\alpha_2\\ 1\ 4\end{pmatrix} - im\begin{pmatrix}\alpha_1\alpha_2\\ 2\ 3\end{pmatrix},\\
v_0 &:= (\xi\varepsilon\gamma_0\eta) = im\begin{pmatrix}\alpha_1\alpha_2\\ 1\ 4\end{pmatrix} - im\begin{pmatrix}\alpha_1\alpha_2\\ 2\ 3\end{pmatrix}.
\end{aligned} \tag{5.7}
$$

Coordinates z_μ of the complex Minkowski space (see (5.1)) obtain a simple form

$$z_\mu = \lambda \frac{v_\mu}{s+p}. \tag{5.8}$$

Relation (5.5) now takes the form

$$s^2 - p^2 - v_\mu v^\mu = 0. \tag{5.9}$$

Recall that representation (5.6) is adapted to the neighborhood $m\binom{\alpha_1\alpha_2}{1\ 2} = \frac{1}{2}(s+p) \neq 0$ and in other neighborhoods, other representations must be used. Nevertheless, formulae (5.8) and (5.9) are independent of the representation. This becomes clear when one considers transformation properties with respect to the group SU(2,2). Let us consider an arbitrary representation of γ-matrices satisfying

$$
\begin{aligned}
&\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \quad \gamma_\mu^+ = \gamma_\mu,\\
&\gamma_4 = i\gamma_0, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4
\end{aligned} \tag{5.10}
$$

(μ,ν=1,2,3,4) together with a skew-symmetric matrix ε satisfying

$$\varepsilon^T = -\varepsilon, \quad \varepsilon^2 = -1, \quad (\varepsilon\gamma_\mu)^T = -\varepsilon\gamma_\mu. \tag{5.11}$$

The infinitesimal generators of SU(2,2) in $\mathbf{C}^4$ can be rewritten in terms of the 4×4 matrices (5.10), as

$$d = \frac{-i}{2}\gamma_5, \quad p_\mu = i\lambda^{-1}\gamma_-\gamma_\mu, \quad k_\mu = -i\lambda\gamma_+\gamma_\mu, \quad m_{\nu\mu} = \frac{i}{4}[\gamma_\mu,\gamma_\nu], \quad \gamma_\pm = \frac{1}{2}(1\pm\gamma_5). \tag{5.12}$$

It is easy to check that these generators satisfy the well known commutation relations of the Lie algebra of SU(2,2)

$$
\begin{aligned}
&[d,p_\mu]=ip_\mu, \quad [d,k_\mu]=-ik_\mu, \quad [d,m_{\mu\nu}]=0,\\
&[p_\mu,p_\nu]=[k_\mu,k_\nu]=0, \quad [k_\mu,p_\nu]=2ig_{\mu\nu}d-2im_{\mu\nu},\\
&[m_{\mu\nu},p_\lambda]=-ig_{\mu\lambda}p_\nu+ig_{\nu\lambda}p_\mu,\\
&[m_{\mu\nu},k_\lambda]=-ig_{\mu\lambda}k_\nu+ig_{\nu\lambda}k_\mu,\\
&[m_{\mu\nu},m_{\rho\lambda}]=i\{g_{\nu\rho}m_{\mu\lambda}+g_{\mu\lambda}m_{\nu\rho}-g_{\mu\rho}m_{\nu\lambda}-g_{\nu\lambda}m_{\mu\rho}\}.
\end{aligned}
\tag{5.13}
$$

Let us consider six forms

$$
s=(\xi\varepsilon\gamma_5\eta), \quad p=(\xi\varepsilon\gamma_5\eta), \quad v_\mu=(\xi\varepsilon\gamma_\mu\eta),
$$

with some factors $\xi,\eta\in\mathbf{C}^4$, prevailing the arbitrariness of representation of the γ's. The action of the generators (5.12) in $\mathbf{C}^4$ induces the following action of these generators on the bilinear asymmetric forms (5.12):

$$
\begin{array}{lll}
ds=ip & dp=is & dv_\mu=0\\
p_\mu s=i\lambda^{-1}v_\mu & p_\mu p=i\lambda^{-1}v_\mu & p_\mu v_\lambda=i\lambda^{-1}g_{\mu\lambda}(s+p)\\
k_\mu s=i\lambda v_\mu & k_\mu p=i\lambda v_\mu & k_\mu v_\lambda=i\lambda^{-1}g_{\mu\lambda}(s-p)\\
m_{\mu\nu}s=0 & m_{\mu\nu}p=0 & m_{\mu\nu}v_\lambda=-ig_{\mu\lambda}v_\nu+ig_{\nu\lambda}v_\mu.
\end{array}
\tag{5.14}
$$

These transformation properties imply that a vector transforming properly under rotations must necessarily be of the following form

$$
z_\mu=f(s,p)v_\mu,
$$

where $f(s,p)$ is a function of scalar s and pseudoscalar p. This function can be determined by a further demand that the quantities z_μ transform properly under translations:

$$
z_\mu\rightarrow z'_\mu=z_\mu+a_\mu=(1+ia^\nu p_\nu)z_\mu \;\Leftrightarrow\; p_\nu z_\mu=-ig_{\mu\nu}.
$$

Applying p_μ to $z_\mu=f(s,p)v_\mu$ we find that

$$
p_\mu z_\nu=i\lambda^{-1}\left(\frac{\partial f}{\partial p}-\frac{\partial f}{\partial s}\right)v_\mu v_\nu-\lambda^{-1}g_{\mu\nu}f(s,p)(s+p)=-ig_{\mu\nu}
$$

is satisfied only if

$$
\frac{\partial f}{\partial p}=\frac{\partial f}{\partial s}, \quad f(s,p)=\frac{\lambda}{s+p}.
$$

Notice that only translations and rotations are necessary to determine the form (5.8) in an arbitrary representation (5.10), (5.11) of the γ-matrices. The behavior with respect dilations and special conformal transformations follows as a consequence of (5.8). The complete set of infinitesimal transformations of z_μ under the full conformal group is

$$
dz_\mu=-iz_\mu, \quad p_\mu z_\lambda=-ig_{\mu\lambda}, \quad k_\mu z_\lambda=+ig_{\mu\lambda}z_\nu z^\nu-2iz_\mu z_\lambda, \quad m_{\mu\nu}z_\lambda=-ig_{\mu\lambda}z_\nu+ig_{\nu\lambda}z_\mu.
\tag{5.15}
$$

Thus, dilation d and rotations $m_{\mu\nu}$ are linear transformations, and, therefore, they act in the same way on the real and on the imaginary parts of the complex vector $z_\mu = x_\mu + iy_\mu$. Special conformal transformations are non-linear and, therefore, they mix x_μ and y_μ according to

$$\begin{aligned} k_\mu x_\lambda &= ig_{\mu\lambda}(x_\nu x^\nu - y_\nu y^\nu) - 2i(x_\mu x_\lambda - y_\mu y_\lambda), \\ k_\mu y_\lambda &= 2ig_{\mu\lambda}x_\nu y^\nu - 2i(x_\mu y_\lambda + y_\mu x_\lambda), \end{aligned} \tag{5.16}$$

Action of translations is also non-linear and (5.15) implies that the real part x_μ transforms like a vector $p_\mu x_\lambda = -ig_{\mu\lambda}$ whereas the imaginary part is translation invariant $p_\mu y_\lambda = 0$.

Transformation properties (5.15) prove that the condition of consistency of the projection (5.8) with the group is satisfied for the complex vector $z_\mu = x_\mu + iy_\mu$. The real and imaginary parts of $z_\mu = x_\mu + iy_\mu$ transform like vectors with respect to rotations and dilations. The fact that y_μ is invariant under translations, and x_μ transforms like a vector, suggests the interpretation of x_μ as the local coordinates of the center of mass and of y_μ as relative coordinates (coordinate differences) with respect to the center of mass. This interpretation corresponds to Yukawa's idea of bilocal theory.[10,11]

Relation (5.9) between the forms (5.7) provides a simple geometrical interpretation of the projection (5.8). Dividing (5.8) and (5.9) by s in the neighborhood $s \neq 0$ we obtain

$$z_\mu = \lambda \frac{v_\mu / s}{1 + p/s}, \quad \frac{v_\mu}{s} \cdot \frac{v^\mu}{s} + \frac{p^2}{s^2} = 1. \tag{5.17}$$

This suggests that formula (5.8) can be viewed as a complex stereographic projection of the complex hyperboloid (5.9) on the complex Minkowski space. SU(2,2)-transformations of the variables $\xi, \eta \in \mathbf{C}^4$ induce pseudoorthogonal $O(4,2)$-transformations of the bilinear skew-symmetric forms s, p, and v_μ (5.7). The last induce conformal transformations of the complex Minkowski coordinates z_μ by the intermediary of the complex stereographic projection of the complex hyperboloid (5.17) in the complex projective space $\mathbf{C}P^5$ with local coordinates p/s, v_μ/s, $\mu = 0,1,2,3$, onto the complex Minkowski space.

Let us go over now to the calculation of invariants of the theory. According to (4.13–14), we have to calculate invariants I_l for the particular metric tensors of SU(2,2) and SU(m) in $\mathbf{C}^{4m}$ and for the particular manifold $\mathscr{O}_2^{4m}$, i.e., for $k=2$, $l=1,2$. We shall use a mixed coordinate system consisting of the elements of the matrices K, a, B. Introducing the $2\times m$ matrix

$$C = (K|B) = \{m_{a\alpha}\}_{\alpha=1,\ldots,m}^{a=1,2} \tag{5.18}$$

we can write

$$I_l = \mathrm{Tr}(F_2 C^* f_1 C)^l, \quad l = 1,2. \tag{5.19}$$

The metric for the group SU(m) is $F_2 = \mathbb{1}_m$. The invariant form of the group SU(2,2) is $(\xi^*, \gamma_4 \eta)$, as one can easily check by applying the generators (5.12) (note that $X_k(\xi\gamma\eta)^* = -(X_k(\xi\gamma\eta)^*)$. In representation (5.6) we have

$$F_1 = \gamma_4 = -\begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} \tag{5.20}$$

and the invariants take, on $\mathscr{O}_2^{4m}$, the form

$$I_l = \mathrm{Tr}(f_1 C C^*)^l, \quad l = 1,2, \tag{5.21}$$

where

$$f_1=a^*F_1a=-(a^*+a). \tag{5.22}$$

Note that the right hand side of (5.22) the original (2.8) (and not the extended) matrix a appears. With the help of homeomorphism (5.1) we can express the complex matrix a through the complex vector z_μ

$$-f_1=a^*+a=-\frac{1}{i\lambda}\tilde{\sigma}^\mu(z_\mu^*-z_\mu)=\frac{2}{\lambda}y_\mu\tilde{\sigma}^\mu \tag{5.23}$$

and obtain

$$I_1=-\frac{2}{\lambda}y_\mu \mathrm{Tr}\tilde{\sigma}^\mu CC^*,$$

$$I_2=\frac{4}{\lambda^2}y_\mu y_\nu \mathrm{Tr}\tilde{\sigma}^\mu CC^*\tilde{\sigma}^\nu CC^*. \tag{5.24}$$

We denote

$$r_\mu=-\mathrm{Tr}\tilde{\sigma}^\mu CC^*=-\sum_{a=1}^{2}\sum_{\alpha=1}^{m} m_{\alpha a}^*(\tilde{\sigma}_\mu)^{\dot{a}b}m_b. \tag{5.25}$$

One can show that

$$\mathrm{Tr}\tilde{\sigma}_\mu CC^*\tilde{\sigma}_\nu CC^*=\sum_{a,b,c,d=1}^{2}\sum_{\alpha,\beta=1}^{m} m_{\alpha a}^*\tilde{\sigma}_\mu^{\dot{a}b}m_{b\beta}m_{\beta c}^*\tilde{\sigma}_\nu^{\dot{c}d}m_{d\alpha}=-\frac{1}{2}g_{\mu\nu}r_\lambda r^\lambda+r_\mu r_\nu. \tag{5.26}$$

Therefore the two invariants are in this case

$$I_1=\frac{2}{\lambda}y_\mu r^\mu,\quad I_2=\frac{4}{\lambda^2}\left\{-\frac{1}{2}y_\mu y^\mu r_\nu r^\nu+(y_\mu r^\mu)^2\right\}. \tag{5.27}$$

Instead of the invariants I_1 and I_2 we can use, equivalently, the invariants $y\cdot r=y_\mu r^\mu$, and $|y|^2\cdot|r|^2=y_\mu y^\mu r_\nu r^\nu$ and describe the decomposition of $\mathscr{O}^2_{(4,m)}$ into a two-parametric family of submanifolds by the two $\mathrm{SU}(2,2)\times\mathrm{SU}(m)$-invariant equations

$$y_\mu r^\mu=-c_{12},\quad y_\mu y^\mu r_\nu r^\nu=c_1, \tag{5.28}$$

with some constants c_1 and c_{12}. We already know the transformation properties or y_μ with respect to SU(2,2) (cf., (5.15–16)). From these transformation properties it follows that $y_\mu y^\mu$ is Poincaré invariant. Let us now derive the transformation properties of r_μ. For this purpose we rewrite r_μ in terms of the γ-matrices in representation (5.6). According to definition (5.25) and notation (5.4), we have

$$r_\mu=-\sum_{a,b=1}^{2}\sum_{\alpha=1}^{m} m_{\alpha a}^*(\tilde{\sigma}_\mu)^{\dot{a}b}m_{b\alpha}=i(\xi^*\gamma_4\gamma_\mu\gamma_+\xi). \tag{5.29}$$

Applying now the infinitesimal SU(2,2) generators (5.12), we obtain

$$\begin{gathered} dr_\mu=ir_\mu,\quad p_\mu r_\lambda=0,\quad m_{\mu\nu}r_\lambda=-ig_{\mu\lambda}r_\nu+ig_{\nu\lambda}r_\mu, \\ k_\mu r_\lambda=2i\{g_{\mu\lambda}r_\nu x^\nu+x_\mu r_\lambda-r_\mu x_\lambda-\varepsilon_{\mu\lambda\rho\nu}r^\rho y^\nu\}. \end{gathered} \tag{5.30}$$

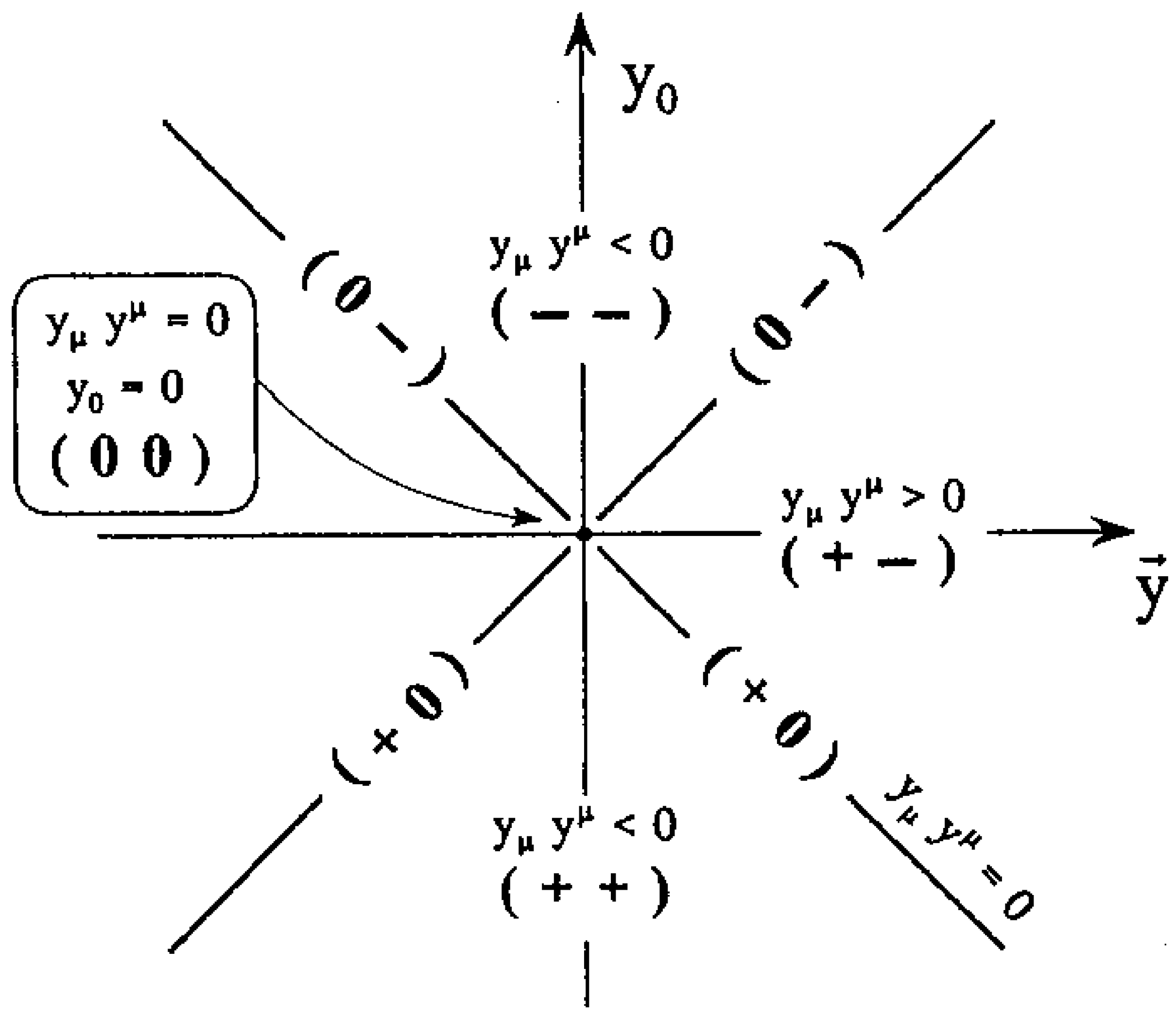

FIG. 6. Six label regions corresponding to six types of induced metric structure.

It follows that

$$p_\mu r_\lambda r^\lambda = m_{\mu\nu} r_\lambda r^\lambda = 0 \tag{5.31}$$

in accordance with (5.2).

Further reduction of the SU(2,2) symmetry to Poincaré symmetry causes appearance of another invariant, namely the Poincaré invariant $r_\lambda r^\lambda$. Consequently, there is a further decomposition of $\mathscr{O}^2_{(4,m)}$ into a three parameter family of submanifolds $\mathscr{O}^{2c}_{(4,m)}$ described by the equations

$$y_\mu y^\mu + c_{11} = y_\mu r^\mu + c_{12} = r_\mu r^\mu + c_{22} = 0. \tag{5.32}$$

Let us consider finally the decomposition of the Grassman manifold $\mathscr{G}^4_2$ into subdomains corresponding to different induced metrics. According to (4.18), the induced signatures are determined by the roots of the secular equation (4.15). In our case (cf., (5.20), (5.23))

$$f_1 = -\frac{2}{\lambda} y_\mu \tilde{\sigma}^\mu,$$

$$\lambda_{1,2} = \frac{1}{2}\mathrm{Tr} f_1 \pm \sqrt{\left(\frac{1}{2}\mathrm{Tr}\, f_1\right)^2 - \det f_1}, \tag{5.33}$$

$$\det f_1 = -\frac{4}{\lambda^2} y_\mu y^\mu, \quad \mathrm{Tr}\, f_1 = -\frac{4}{\lambda} y_0.$$

According to the general scheme (4.20) and Figure 4, we have the following six domains corresponding to the six admissible metric types $(++)$, $(+-)$, $(--)$, $(+0)$, $(0-)$, (00) (see Figure 6):

$$
\begin{array}{lllll}
(++) & y_0<0 & y_\mu y^\mu<0 & \lambda_1>0 & \lambda_2>0, \\
(+\,0) & y_0<0 & y_\mu y^\mu=0 & \lambda_1=0 & \lambda_2>0, \\
(+-) & & y_\mu y^\mu>0 & \lambda_1<0 & \lambda_2>0, \\
(0\,-) & y_0>0 & y_\mu y^\mu=0 & \lambda_1<0 & \lambda_2=0, \\
(--) & y_0>0 & y_\mu y^\mu<0 & \lambda_1<0 & \lambda_2<0, \\
(0\,0) & y_0=0 & y_\mu y^\mu=0 & \lambda_1=0 & \lambda_2=0.
\end{array}
\tag{5.34}
$$

VI. INTERNAL STRUCTURE

We have, so far determined the decomposition of $\mathscr{O}^2_{(4,m)}$ into $SU(2,2)\times SU(m)$ and $P_4\times SU(m)$ invariant submanifolds $\mathscr{O}^{2C}_{(4,m)}$. To give a common description of both cases, we introduce the shorthand notation

$$c_{11}=-y_\mu y^\mu, \quad c_{12}=-y_\mu r^\mu, \quad c_{22}=-r_\mu r^\mu, \tag{6.1}$$

corresponding to condition (5.28), (5.32). In the conformally invariant case only c_{12} and $c_1=c_{11}c_{22}$ can be considered as constants. In the Poincaré-invariant case, all three quantities c_{11}, c_{12}, c_{22} can be considered as constants. Rewriting the second equation of (6.1) in the form

$$y_0=r_0^{-1}(\vec{y}\vec{r}+c_{12}) \tag{6.2}$$

and substituting it into the first, we obtain a second order equation for the coordinates y_1,y_2,y_3, of the vector $\vec{y}$ with coefficients depending on r_μ, $\mu=0,1,2,3$, and the constants c_{12}, c_1:

$$\vec{y}^2-\frac{(\vec{r}\vec{y}+c_{12})^2}{r_0^2}=\frac{c_1}{r_\mu r^\mu}. \tag{6.3}$$

By a proper transformation of the coordinates we can bring this equation into diagonal form

$$(y_1')^2+(y_2')^2+\frac{c_{22}}{r_0^2}(y_3')^2=-\frac{\det c}{c_{22}}, \tag{6.4}$$

which represents various types of second order surfaces in three dimensional space depending on the values of the coefficients appearing in (6.4).

To get better insight to the situation, we calculate the quantity $c_{22}=-r_\mu r^\mu$ in terms of the variables $m_{a\alpha}$, $a=1,2$, $\alpha=1,..,m$, (cf., (5.25)). The result is

$$c_{22}=-r_\mu r^\mu=2\sum_{\alpha_1,\alpha_2=1}^{m}\left|m\begin{pmatrix}\alpha_1 & \alpha_2\\ 1 & 2\end{pmatrix}\right|^2. \tag{6.5}$$

In the case of $SU(m)$ internal symmetry $(q=0)$, c_{22} is always positive. The value $c_{22}=0$ is excluded on $\mathscr{O}_2^{(4,m)}$ in domains $m\binom{\alpha_1\alpha_2}{1\;2}\neq 0$ which we consider. Also

$$r_0=\sum_{a=1}^{2}\sum_{\alpha=1}^{m}|m_{a\alpha}|^2>0$$

is strictly positive. Due to these facts, the quadratic form (6.4) is positive definite and the condition that the surface is real is

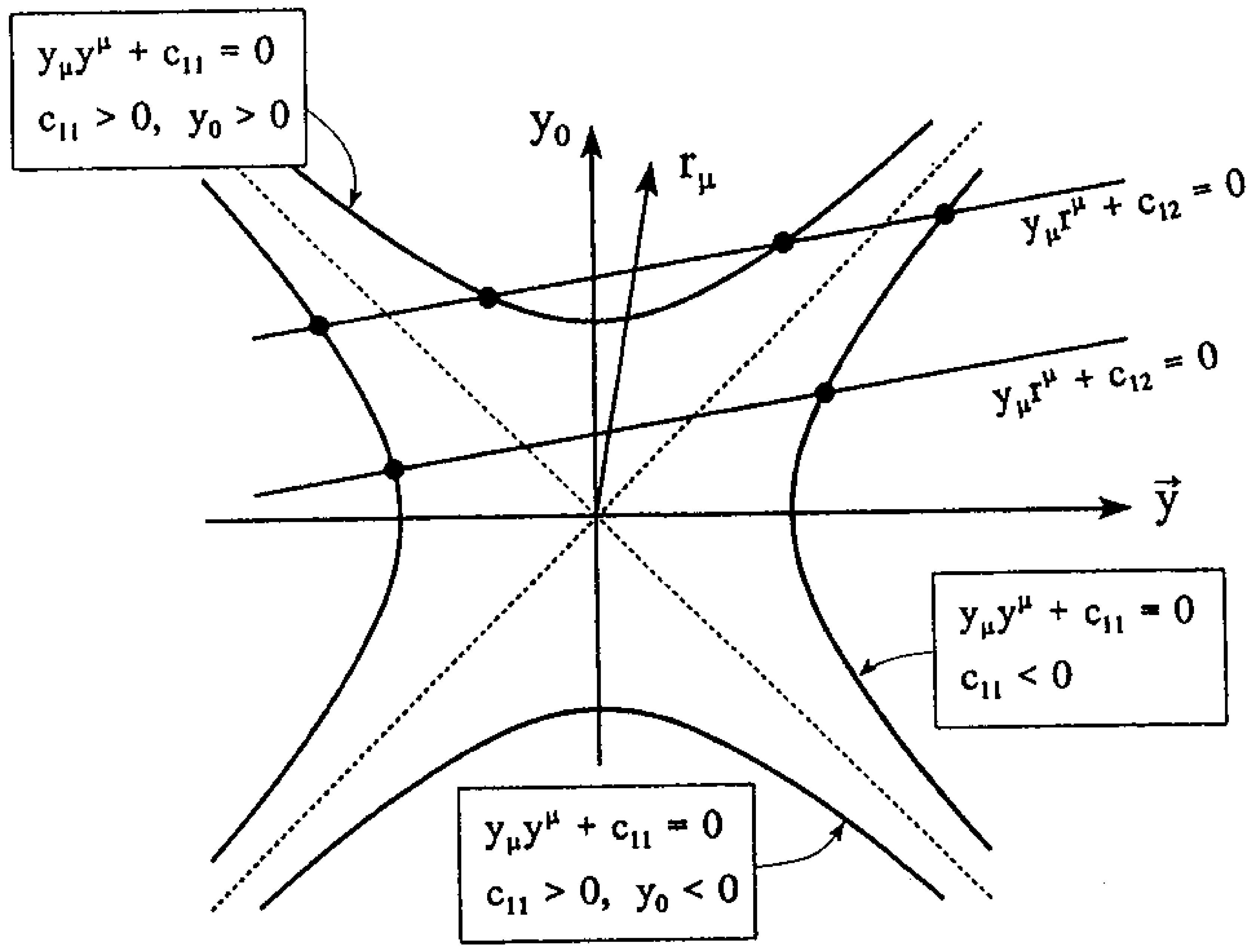

FIG. 7. Different values of the invariant **c**.

$$\det c = c_{11}c_{22} - c_{12}^2 = c_1 - c_{12}^2 < 2. \tag{6.6}$$

Otherwise we have to do with disjoint surfaces in the space of variables y_μ, $\mu=0,1,2,3$ (see Figure 7). Let us note that det c is always negative when y_μ is space-like ($c_{11}<0$).

We can solve equations (5.28), (5.32) with respect to $y=\sqrt{y_1^2+y_2^2+y_3^2}$ or $r=\sqrt{r_1^2+r_2^2+r_3^2}$ by eliminating y_0 or r_0 from the equations (cf., (6.1))

$$y^2 - y_0^2 + c_{11} = yr\,\cos\theta - y_0 r_0 + c_{12} = r^2 - r_0^2 + c_{22}. \tag{6.7}$$

The result is

$$y = \frac{c_{12} r\,\cos\theta \pm r_0\sqrt{-\det c - c_{11} r^2\,\sin^2\theta}}{c_{22} + r^2 \sin^2\theta},$$

$$r = \frac{c_{12} y\,\cos\theta \pm y_0\sqrt{-\det c - c_{22} y^2\,\sin^2\theta}}{c_{11} + y^2\,\sin^2\theta}, \tag{6.8}$$

with the reality conditions for the roots

$$\det c + c_{11} r^2 \sin^2\theta < 0 \quad \text{or} \quad \det c + c_{22} y^2 \sin^2\theta < 0. \tag{6.9}$$

The situation is particularly simple for the vector $r_i=0$, $r_0=\sqrt{c_{22}}$. In this case (cf., Eqs. (6.4), (6.8))

$$y = \sqrt{-\frac{\det c}{c_{22}}}$$

is an equation of a sphere S^2 with radius $\sqrt{-\det c/c_{22}}$. It is important to have also a coordinate free description of the spaces determined by the conditions (5.28) or (5.32) together with equations (5.25), (5.29). Here, we shall restrict ourselves to the Poincaré invariance. In this case one can show that the submanifolds $\mathcal{O}_{2,c}^{(4,m)}$ are direct products of a real Minkowski space M_4 and a homogeneous space M^{int} of the group $SU(3,1)\times SU(m)$. To convince ourselves of this fact, we consider a point with coordinates $\mathring{m}_{a\alpha}$, $\mathring{y}_\mu$ satisfying the conditions

$$\mathring{y}_1=\mathring{y}_2=0, \quad \mathring{m}_{1\alpha}=0, \quad \mathring{m}_{2\alpha'}=0 \tag{6.10}$$

for $\alpha=2,3,\ldots,m$ and $\alpha'=3,4,\ldots,m$, and

$$\begin{gathered}\mathring{r}_i=-\mathring{m}^*_{\alpha a}(\tilde{\sigma}_i)^{ab}\mathring{m}^*_{b\alpha}=0, \quad \mathring{r}_0=-\mathring{m}^*_{\alpha a}(\tilde{\sigma}_0)^{ab}\mathring{m}^*_{b\alpha}=\sqrt{c_{22}},\\ (\mathring{y}_3)^2-(\mathring{y}_0)^2=-c_{11}, \quad \mathring{y}_0\mathring{r}_0=c_{12}, \quad \det c<0.\end{gathered} \tag{6.11}$$

This point satisfies conditions (5.32) and the isotropy group of this point is $SO(2)\times SU(m-2)$. Moreover, every point satisfying (5.32) can be reached from the point (6.10) by a transformation of $SO(3,1)\times SU(m)$. We can write therefore, globally,

$$\mathcal{O}_{2,c}^{(4,m)}=\frac{P_4}{SO(3,1)}\times\frac{SO(3,1)\times SU(m)}{SO(2)\times SU(m-2)}, \tag{6.12}$$

where c denotes the three real parameters c_{ik} satisfying $\det c<0$. The first quotient $M_4=P_4/SO(3,1)$ represents the real Minkowski space M_4 parameterized by coordinates $x_\mu=Re z_\mu$. This space is not affected by the invariance conditions (5.32) and it transforms into itself under dilations d, translations p_μ, and rotations $m_{\mu\nu}$. The internal space

$$M^{\text{int}}=\frac{SO(3,1)\times SU(m)}{SO(2)\times SU(m-2)} \tag{6.13}$$

can be viewed as the direct product of five-dimensional outer internal space SO(3,1)/SO(2) parameterized by coordinates y_μ, r_μ subject to conditions (5.32) which does not depend on m and is invariant with respect to SU(m) and the inner internal space $SU(m)/SU(m-2)$. Both spaces are translation invariant.

In the case of the larger SU(2,2)-symmetry, the situation is more complicated due to the fact that special conformal transformations mix the variables x_μ, y_μ, r_μ (cf., e.g., (5.16)). The domains described by the admissible metrics $(++)$, $(+-)$, $(--)$ correspond to $c_{11}\neq 0$. The degenerate metrics $(+\,0)$, $(0\,-)$, and $(0\,0)$ correspond to $c_{11}=0$. The last one involves the additional condition $y_0=0$ so that the internal space reduces to one point.

One can give also a local geometrical interpretation of the internal space M^{int}. Indeed, $SO(3,1)\times SO(2)$ can be considered locally as the direct product of a three-dimensional hyperboloid H^3 and a two-dimensional sphere S^2. Vector r_μ moves on H^3 while y_μ moves on a 2-dimensional ellipsoid (see Figure 7). Similarly, vectors $m_{1\alpha}$ and $m_{2\alpha}$, $\alpha=1,\ldots,m$, move on a $(2m-1)$-dimensional sphere and on a sphere with two dimensions less, i.e. on S^{2m-3}, respectively (the second vector is additionally restricted by the length and the scalar product). We can consider, therefore, M^{int} locally as the direct product

$$M^{\text{int}}=S^2\times H^3\times S^{2m-1}\times S^{2m-3}. \tag{6.14}$$

Thus, the internal space in this $P_4\times SU(m)$ invariant particle model is not compact due to the appearance of the homogeneous manifold SO(3,1)/SO(2) in (6.13) or the hyperboloid H^3 in (6.14). The other factors $SU(m)\times SU(m-2)$ in (6.13), or $S^2\times S^{2m-1}\times S^{2m-3}$ in (6.14), are compact. Moreover, there is one factor which is independent on the number m and which has, therefore, a universal meaning.

For full physical interpretation, it remains to find the representations $SU(2,2)\times SU(m)$ and $P_4\times SU(m)$ in the corresponding homogeneous spaces. We shall present the results in a separate publication.

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