The geometry of elastic collisions and Heron's law

Jerzy Kocik[a)]
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

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Elastic collision of two balls on a line is discussed in terms of their configuration space. The optical-mechanical analogy is analyzed in this context. In particular, the law of collision is reinterpreted as Heron's law of light reflection.

I. INTRODUCTION

One of the standard tricks used in describing physical systems is construction of a *configuration space*. By definition, a *point* in a configuration space corresponds to a configuration of the system. The basic example concerns the many-body problem: k particles in $\mathbf{R}^3$ may equivalently be represented by one "particle" in $\mathbf{R}^{3n}$. Here $\mathbf{R}^3$ is a model of the "visual space," $\mathbf{R}^{3n}$ is the "configuration space." Thus the low dimensionality of the visual space is traded for simplicity (one point!).

In the following, we give a geometric description of the elastic collision of two particles on a line. Despite its simplicity, this example illustrates the idea of "dimensional blow-up," provides an interesting example of an optical-mechanical analogy, and illustrates the role of Euclidean geometry in classical mechanics.

A matrix approach to analysis of elastic collisions was introduced by Romer,[1] and was employed later for related problems.[2,3] Other aspects of geometry of elastic collisions are also investigated in Refs. 4–6.

II. ELASTIC COLLISION

Consider two particles on a line $\mathbf{R}$. The configuration space is a two-dimensional plane $\mathbf{R}^2$. If x_A and x_B denote positions of the first and second particles, then, in $\mathbf{R}^2$, the pair (x_A, x_B) stands for the single point (say, a *biparticle*) that represents the system. Since particles cannot occupy the same position at the same time, the diagonal set $\Delta = \{(x_A, x_B) | x_A = x_B\}$ must be excluded from the configuration space.

We shall see that the *elastic collision* of two particles in $\mathbf{R}$ corresponds to *reflection* from the set Δ in $\mathbf{R}^2$. Indeed, if two particles have the same mass, a collision is just an exchange of velocities (see Fig. 1). That is to say, if v_A and v_B are the initial velocities on $\mathbf{R}$, then the after-collision velocities are $w_A = v_B$ and $w_B = v_A$. In the configuration space $\mathbf{R}^2$, this can be viewed as a reflection of a velocity vector $\mathbf{v} = (v_A, v_B)$ from the "mirror" Δ with resulting vector $\mathbf{w} = (w_A, w_B)$. This reflection of velocity is a linear transformation $T\mathbf{v} = \mathbf{w}$, that in coordinates translates into:[7]

$$\begin{bmatrix} w_A \\ w_B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix}. \tag{2.1}$$

As a motivational exercise, the reader is asked to interpret the events shown in Fig. 2.

According to the *law of angles* for reflection of light in a mirror, the angle of incidence equals the angle of reflection. It is interesting that in the construction of configuration space described above, a law of optics seems to replace laws of mechanics. One may ask how far this analogy goes. In particular, if the masses of the particles differ, is the optical analogy still valid for elastic collision?

Remark: It was Heron of Alexandria who in the first century derived the "law of angles" from the minimum principle—the position of the point of reflection from a mirror minimizes the length of the light path from the source to the point of arrival (his proof is purely geometrical). This historical fact is well-known among mathematicians[8] but is hardly acknowledged within the lore of the history of physics,[9] despite the fact that it has served Fermat as the motivation for his "minimum principle" for optics, which later led to Maupertuis' principle,[10] Hamilton's method of characteristics, etc., becoming the source and the prototype of the variational calculus in physics.

In the following, the law of equal angles for mirrors will be addressed simply as *Heron's law*.

III. COLLISION MATRIX

Throughout this section we use the following notation:

	Particle A	Particle B
Mass	m_A	m_B
Initial velocity	v_A	v_B
Final velocity	w_A	w_B

We know that, given masses and initial velocities, the final velocities after an elastic collision are determined by the laws of energy and of momentum conservation. In our case of a one-dimensional system, from

(i) energy conservation: $m_A v_A^2 + m_B v_B^2 = m_A w_A^2 + m_B w_B^2$,

(ii) momentum conservation: $m_A v_A + m_B v_B = m_A w_A + m_B w_B$ (3.1)

one gets the result:

$$w_A = \frac{(m_A - m_B) v_A + 2 m_B v_B}{m_A + m_B},$$
$$w_B = \frac{(m_B - m_A) v_B + 2 m_A v_A}{m_A + m_B}. \tag{3.2}$$

Notice that this is a linear transformation and can be written in matrix form (cf. Ref. 1)

$$\begin{bmatrix} w_A \\ w_B \end{bmatrix} = \frac{1}{m_A + m_B} \begin{bmatrix} m_A - m_B & 2m_B \\ 2m_A & m_B - m_A \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix},$$

where

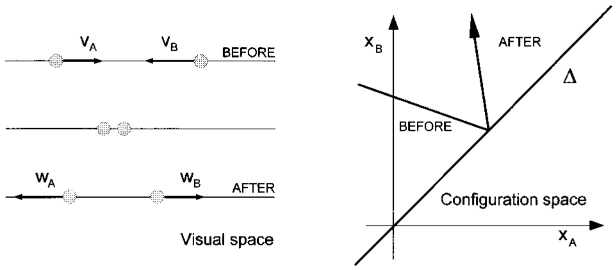

Fig. 1. A two-particle system on a line (A) as a single particle in a plane (B).

$$T=\begin{bmatrix} \dfrac{m_A-m_B}{m_A+m_B} & \dfrac{2m_B}{m_A+m_B} \\ \dfrac{2m_A}{m_A+m_B} & \dfrac{m_B-m_A}{m_A+m_B} \end{bmatrix} \tag{3.3}$$

will be called the *collision matrix*. Notice that T assumes the form of (2.1) if the masses are equal.

Although (3.1) involves quadratic equations, one can derive (3.2) without leaving the domain of linear algebra. We start by introducing the following:

Law 1: (*Law of averaging velocities*) Elastic collision of two balls makes their average velocities equal:

$$v_A+w_A=v_B+w_B\,. \tag{3.4}$$

Proof: Rewrite the law of energy conservation (3.1i) in a form of square differences:

$$m_Av_A^2-m_Aw_A^2=m_Bw_B^2-m_Bv_B^2.$$

This may be expressed in terms of products

$$m_A(v_A-w_A)(v_A+w_A)=m_B(w_B-v_B)(v_B+w_B).$$

In the general case, one may cancel out the law of momentum conservation (3.1ii) (also in the form of differences) and get (3.4). If $v_A-w_A=0$ (and consequently $v_B-w_B=0$), then such a cancellation is invalid. However, in this case, collision cannot occur: Indeed, $v_A=w_A$ and $v_B=w_B$ imply that the trajectory forms a straight line, which must be parallel to Δ [see Fig. 2] to avoid passing through it. But then $v_A=w_A=v_B=w_B$ holds all the time, and (3.4) is trivially satisfied. □

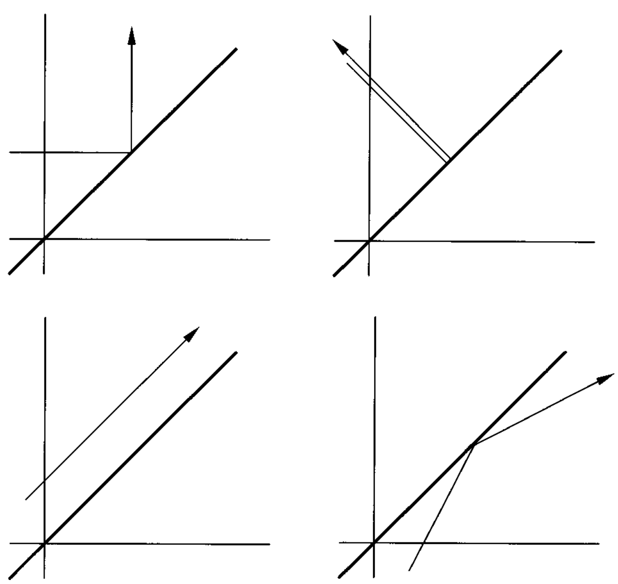

Fig. 2. Heron's law of elastic collisions.

Now, a derivation of the formula for elastic collision (3.2) becomes painless. Let us first restate it in matrix form.

Law 2: (*Elastic collision*) If vectors $\mathbf{v}=(v_A,v_B)$ and $\mathbf{w}=(w_A,w_B)$ represent velocities of two particles in $\mathbf{R}$ before and after collision, respectively, then

$$\mathbf{w}=T\mathbf{v}, \tag{3.5}$$

where the matrix T is

$$T=\begin{bmatrix} \dfrac{m_A-m_B}{m_A+m_B} & \dfrac{2m_B}{m_A+m_B} \\ \dfrac{2m_A}{m_A+m_B} & \dfrac{m_B-m_A}{m_A+m_B} \end{bmatrix}.$$

Proof: The law of averaging velocities (3.4) reduces the problem to a system of two linear equations that may be solved almost instantly. Indeed, recall

velocity averaging: $w_A-w_B=v_A-v_B\,,$

momentum conservation: $m_Av_A+m_Bv_B=m_Aw_A+m_Bw_B\,,$

which may be written in matrix form:

$$\begin{bmatrix} 1 & -1 \\ m_A & m_B \end{bmatrix}\begin{bmatrix} w_A \\ w_B \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ m_A & m_B \end{bmatrix}\begin{bmatrix} v_A \\ v_B \end{bmatrix}.$$

Since the left-hand-side matrix is nonsingular, one can write

$$\begin{bmatrix} w_A \\ w_B \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ m_A & m_B \end{bmatrix}^{-1}\begin{bmatrix} -1 & 1 \\ m_A & m_B \end{bmatrix}\begin{bmatrix} v_A \\ v_B \end{bmatrix}.$$

Thus the collision matrix is

$$\begin{aligned} T&=\begin{bmatrix} 1 & -1 \\ m_A & m_B \end{bmatrix}^{-1}\begin{bmatrix} -1 & 1 \\ m_A & m_B \end{bmatrix} \\ &=\frac{1}{m_A+m_B}\begin{bmatrix} m_B & 1 \\ -m_A & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ m_A & m_B \end{bmatrix} \\ &=\frac{1}{m_A+m_B}\begin{bmatrix} m_A-m_B & 2m_B \\ 2m_A & m_B-m_A \end{bmatrix} \end{aligned}$$

which ends the proof. □

Remark 1: Notice that if one of the masses is negligible, say $m_A=0$, then the collision matrix does not depend on the other mass and is

$$T=\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Thus, simply $w_B=v_B$ (the heavy particle is not affected by the collision), while $w_A=-v_A+2v_B$ (the velocity of the light particle is inverted in the reference system of the heavy particle).

Remark 2: Notice that the triple of laws (3.1i,ii) and (3.4), i.e., energy conservation, momentum conservation, and velocity averaging, form a system in which any two laws determine the third one. One may imagine a civilization in which Law 1 is the fundamental law of mechanics and obviates the need for the law of momentum conservation (or maybe energy conservation) among fundamental principles. What would the science of mechanics, and consequently *physics*, look like?

Corollary 1: The collision matrix satisfies

$$T \circ T = I, \quad \det T = -1 \quad \operatorname{Tr} T = 0, \tag{3.6}$$

where I stands for identity matrix, and T represents reflection in $\mathbf{R}^2$. (Note that reflection is not necessarily an *orthogonal* operator).

Exercise 1: Find the eigenvalues and eigenvectors of T. Determine a geometric method to find $\mathbf{w}$, given $\mathbf{v}$ and masses m_A and m_B.

IV. HERON'S LAW FOR ELASTIC COLLISION

What happened to the simplicity of Eq. (2.1)? Is the collision matrix T still a "mirror reflection" from the diagonal in $\mathbf{R}^2$?

It is easy to find that the eigenvectors of T are

$$\mathbf{D} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} m_B \\ -m_A \end{bmatrix} \tag{4.1}$$

with eigenvalues 1 and -1, respectively. The vector $\mathbf{D}$ determines the orientation of the mirror Δ. (If the biparticle slides along the mirror, no actual reflection, i.e., change of the velocity, occurs). The other eigenvector, $\mathbf{N}$, with eigenvalue -1, indicates that a system with initial velocity parallel to $\mathbf{N}$ will be reflected from Δ with exactly the opposite velocity. But $\mathbf{N}$ does not appear to be orthogonal to Δ, so Heron's law seems to be violated for nonequal masses. Unless . . .

Definition: Define a metric tensor in $\mathbf{R}^2$ by

$$g = \begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}. \tag{4.2}$$

In terms of coordinates, $(\mathbf{a},\mathbf{b}) = m_A a_A b_A + m_B a_B b_B$ replaces the standard product $\mathbf{a}\cdot\mathbf{b}$.

Corollary 2: The reflection matrix T is orthogonal with respect to the metric g.

Proof: An easy calculation shows that $T^T g T = g$, which is the condition for orthogonality of T. □

Thus Heron's law for elastic collisions is satisfied in the general case, *if* the proper Euclidean structure, that of (4.2), is recognized! [Matrix (4.2) of the metric appears in Romer's analysis[1] as a transformation to "u space"]. However, the form of Corollary 2 may be found to be too abstract—let us see Heron's law more directly. First, notice that the eigenvectors of T are mutually orthogonal, $\mathbf{D} \perp \mathbf{N}$:

$$(\mathbf{D},\mathbf{N}) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} m_B \\ -m_A \end{bmatrix} \right) = m_A m_B - m_b m_A = 0. \tag{4.3}$$

To ensure that T obeys Heron's law, we need to look more closely at the geometry of the event of reflection: (i) that the final velocity $\mathbf{w}$ has the same length as the initial velocity $\mathbf{v}$ (measured in terms of g); (ii) that the component of velocity parallel to the mirror Δ is preserved during the collision; as well as (iii) that the normal part of $\mathbf{v}$ is opposite to the normal part of $\mathbf{w}$. Briefly:

$$\text{(i)} \quad |\mathbf{v}| = |\mathbf{w}|, \quad \text{(ii)} \quad \mathbf{v}_\parallel = \mathbf{w}_\parallel, \quad \text{(iii)} \quad \mathbf{v}_\perp = -\mathbf{w}_\perp \tag{4.4}$$

(see Fig. 3). These geometric requirements will clearly ensure that the angle of incidence α equals the angle of reflection β. But there is more than that. These three conditions are

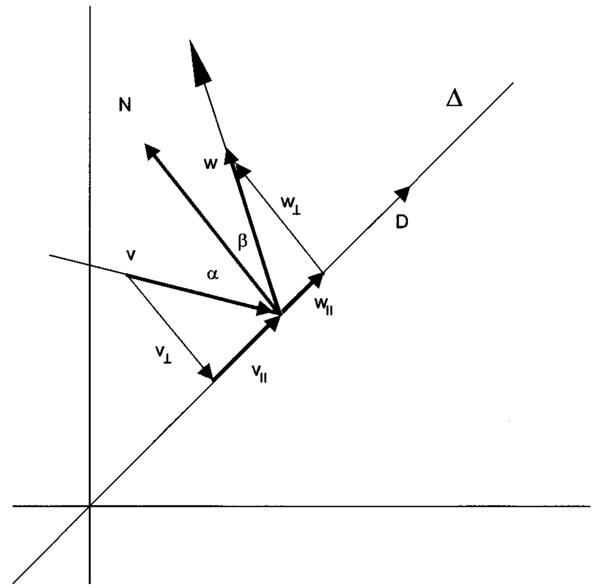

Fig. 3. The geometry of Heron's law of collision.

associated with three images: the mechanics of collision of two masses, the geometry of ricochet of one point off a wall, and the optics of reflection of a ray from a mirror. The three laws of mechanics translate into the trigonometry of reflection, and into Heron's law of optics.

(1) The scalar product (4.2) has the meaning of kinetic energy of the system. In particular, the law of energy conservation (3.1i) translates into

$$(\mathbf{v},\mathbf{v}) = (\mathbf{w},\mathbf{w}). \tag{4.5}$$

Geometrically, this means that the length of incident and reflected vectors are equal in terms of metric g. In optics, this means that the speed of the light ray before and after reflection is the same.

(2) Also, momentum conservation (3.1ii) has a simple geometric meaning. It can be written in the scalar product form:

$$\left(\mathbf{v}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \left(\mathbf{w}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right). \tag{4.6}$$

This says that the tangent velocity, i.e., the component parallel to the mirror surface Δ, is conserved. Trigonometrically, this is Heron's law in the form: $\sin\alpha = \sin\beta$. Thus the law of momentum conservation can be interpreted as a law of optics: "the angle of an incident beam equals the angle of the reflected beam."

Table I. Correspondence between the different representations of collision.

Mechanics Two particles in $\mathbb{R}$	Geometry One particle in $\mathbb{R}^2$	Optics Ray in $\mathbb{R}^2$
Collision	Ricochet	Reflection
masses	metric tensor	transparency
Energy conserved	$(\mathbf{v},\mathbf{v}) = (\mathbf{w},\mathbf{w})$	speed preserved
Momentum conserved	$(\mathbf{v},\mathbf{D}) = (\mathbf{w},\mathbf{D})$	Heron's law ($\sin\alpha = \sin\beta$)
Velocities averaged	$(\mathbf{v},\mathbf{N}) = -(\mathbf{w},\mathbf{N})$	Heron's law ($\cos\alpha = \cos\beta$)
"perfect collision"	eigenvectors	mirror orientation

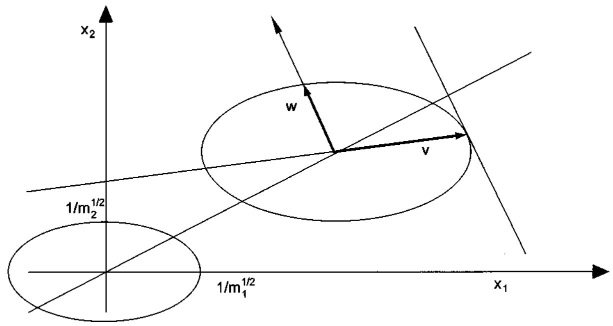

Fig. 4. Geometric construction of the result of collision.

(3) The law of averaging velocities is geometrically dual to that of momentum conservation. Indeed, it can be written as

$$\left(\mathbf{v},\begin{bmatrix} m_B \\ -m_A \end{bmatrix}\right) = -\left(\mathbf{w},\begin{bmatrix} m_B \\ -m_A \end{bmatrix}\right). \tag{4.7}$$

But this is requirement (4.4iii)! The trigonometric version of (4.7) is $\cos\alpha=\cos\beta$, which is just another form of Heron's law.

A table of correspondence between these three pictures can thus be created (see Table I). The ''perfect collision'' in the first column denotes a situation when the particles (or balls) leave the collision preserving their original speeds (with the same or inverted velocities).

Solution to Exercise 1: Here is a simple geometric construction of the final velocity **w** given the initial velocity **v** and masses m_A and m_B (see Fig. 4).

(1) As an initial exercise, draw in R^2 an ellipse that has the main axes along the coordinate axes, with points of intersection $\pm m_A^{1/2}$ along x and $\pm m_B^{-1/2}$ along y. This ellipse represents a unit circle with respect to the metric g. In the optical interpretation, it determines points that would be reached by light propagating from a point (0) in a unit time.

(2) At the collision point of $\mathbf{R}^2$, where the incoming *biparticle* strikes the mirror Δ, draw an ellipse of all vectors of length (in the sense of the metric g) of the incoming vector **v**. This ellipse is similar (rescaled and shifted) to the prototype ellipse drawn at 0.

(3) Determine a line tangent to the ellipse through the end point of vector **v**. Then, draw a parallel line through the center of ellipse.

(4) The vector spanned along this line from the ellipse center to the intersection with the ellipse is the postcollision velocity vector **w**.

Note that by appropriately rescaling the axes in Fig. 4, say $x\to x'=x\sqrt{m_A/m_B}$, the ellipse can be transformed into a circle. Then the visual representation of the general case coincides with the equal-mass case, i.e., the incidence and reflection axes *look* equal, except the diagonal set Δ is skewed, i.e., is now $x'\sqrt{m_B}=y\sqrt{m_A}$.

Notice that the ellipse in the construction in Fig. 4 can be viewed as indicatrix for light propagation.[11]

[a]Electronic mail: jkocik@physics.uiuc.edu

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[7]Note on notation: Vectors are to be understood in the coordinate system as columns, even if they appear as rows in regular text.
[8]R. Courant and H. Robbins, *What is Mathematics* (Oxford U.P., New York, 1941).
[9]Consult, e.g., such basic textbooks as M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1975), or R. K. Luneburg, *Mathematical Theory of Optics* (University of California Press, Berkeley, 1964).
[10]René Dugas, *A History of Mechanics* (Dover, New York, 1988).
[11]For a detailed analysis of the interrelations between Huygens' principle of optics and the Hamilton–Jacobi equation of classical mechanics see V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978), Chap. 9.46.

SIMPLIFICATION

Classical science...works because it simplifies. It takes on only those problems that can be solved by the known method. The entire scientific edifice, for all its hermetic inaccessibility to the uninitiated, is a vast monument to simplification.

Bryan Appleyard, *Understanding the Present—Science and the Soul of Modern Man* (Pan Books, London, 1992), p. 150.