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Apollonian Circle Packings: Geometry and Group Theory
II. Super-Apollonian Group and Integral Packings

Ronald L. Graham
Department of Computer Science and Engineering
University of California at San Diego, La Jolla, CA 92093-0114

Jeffrey C. Lagarias
Department of Mathematics
University of Michigan, Ann Arbor, MI 48109–1109

Colin L. Mallows
Avaya Labs, Basking Ridge, NJ 07920

Allan R. Wilks
AT&T Labs, Florham Park, NJ 07932-0971

Catherine H. Yan [1]
Department of Mathematics
Texas A&M University, College Station, TX 77843-3368

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ABSTRACT

Apollonian circle packings arise by repeatedly filling the interstices between four mutually tangent circles with further tangent circles. Such packings can be described in terms of the Descartes configurations they contain, where a Descartes configuration is a set of four mutually tangent circles in the Riemann sphere, having disjoint interiors. Part I showed there exists a discrete group, the Apollonian group, acting on a parameter space of (ordered, oriented) Descartes configurations, such that the Descartes configurations in a packing formed an orbit under the action of this group. It is observed there exist infinitely many types of integral Apollonian packings in which all circles had integer curvatures, with the integral structure being related to the integral nature of the Apollonian group. Here we consider the action of a larger discrete group, the super-Apollonian group, also having an integral structure, whose orbits describe the Descartes quadruples of a geometric object we call a super-packing. The circles in a super-packing never cross each other but are nested to an arbitrary depth. Certain Apollonian packings and super-packings are strongly integral in the sense that the curvatures of all circles are integral and the curvature$\times$centers of all circles are integral. We show that (up to scale) there are exactly 8 different (geometric) strongly integral super-packings, and that

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each contains a copy of every integral Apollonian circle packing (also up to scale). We show that the super-Apollonian group has finite volume in the group of all automorphisms of the parameter space of Descartes configurations, which is isomorphic to the Lorentz group $O(3,1)$.

Keywords: Circle packings, Apollonian circles, Diophantine equations, Lorentz group, Coxeter group

1. Introduction

Apollonian circle packings are arrangements of tangent circles that arise by repeatedly filling the interstices between four mutually tangent circles with further tangent circles. A set of four mutually tangent circles is called a Descartes configuration. Part I studied Apollonian circle packings in terms of the set of Descartes configurations that they contain. It is observed that there exist Apollonian circle packings that have a very strong integral structure, with all circles in the packing having integer curvatures, and rational centers, such that curvature×center is an integer vector. We termed these *strongly integral* Apollonian circle packings. An example is the $(0,0,1,1)$ packing pictured in Figure 1, with the two circles of radius 1 touching at the origin, and with two straight lines parallel to the x-axis.

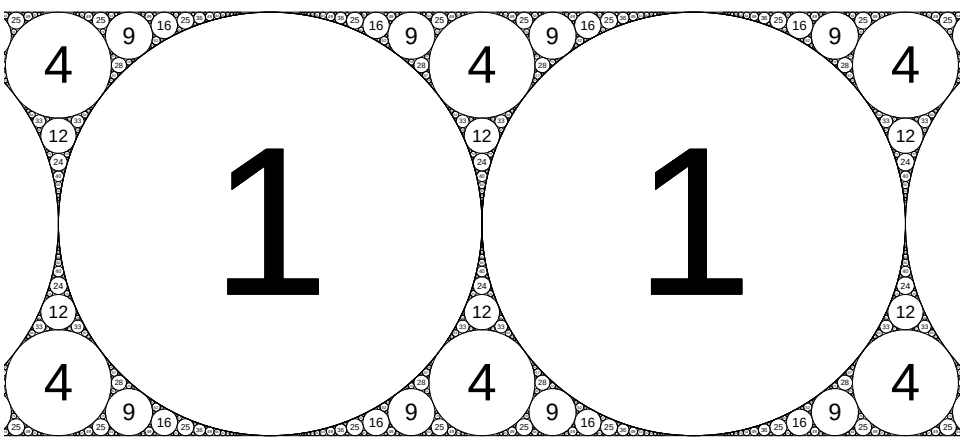

Figure 1: The integer Apollonian packing $(0,0,1,1)$

Part I gave an explanation for the existence of such integral structures. This uses a coordinate system for describing all (ordered, oriented) Descartes configurations $\mathcal{D}$ in terms of their curvatures and centers, which forms a 4×3 *curvature-center coordinate* matrix $\mathbf{M}_{\mathcal{D}}$, and a more detailed coordinate system, *augmented curvature-center coordinates*, using 4×4 matrices $\mathbf{W}_{\mathcal{D}}$. The strongly integral property of a single Descartes configuration is encoded in the integrality of the matrix $\mathbf{M}_{\mathcal{D}}$. The set of all (geometric) Descartes configurations in an Apollonian packing can be described as a single orbit of a certain discrete group $\mathcal{A}$ of 4×4 integer matrices; algebraically there are 48 orbits of ordered, oriented Descartes configurations giving rise to the same geometric packing, which correspond to the 48 possible ways of ordering the circles and totally orienting the configuration. The integrality of the members of $\mathcal{A}$ is the

source of the strong integrality of some Apollonian circle packings. As a consequence of this group action, if a single Descartes configuration in the packing is strongly integral, then they all are; hence every individual circle in the packing is strongly integral.

There are infinitely many distinct integral Apollonian circle packings. Two more of them are pictured in Figures 2 and 3, the $(-1, 2, 2, 3)$-packing and the $(-6, 11, 14, 15)$-packing, respectively. Any integral packing can be moved by a Euclidean motion so as to be strongly integral, as will follow from results in this paper.

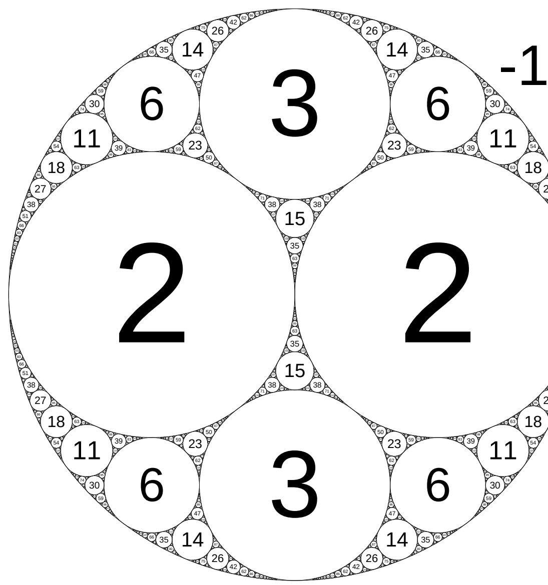

Figure 2: The integer Apollonian packing $(-1, 2, 2, 3)$

Part I introduced some further integral actions on Descartes configurations, involving a "duality" operator $\mathbf{D}$ leading to a dual Apollonian group $\mathcal{A}^{\perp}$. Combining this group with the Apollonian group leads to a large group of integer 4×4 matrices, the *super-Apollonian group* $\mathcal{A}^S$, which acts on the set of all (ordered, oriented) Descartes configurations. Part I showed that the super-Apollonian group is a hyperbolic Coxeter group. It defined an *Apollonian super-packing* to be an orbit $\mathcal{A}^S[\mathcal{D}]$ of a single Descartes configuration $\mathcal{D}$ under this group. Such a super-packing is called *integral* if the initial Descartes configuration has integer curvatures, and is *strongly integral* if the curvature-center coordinates $\mathbf{M}_{\mathcal{D}}$ of the initial Descartes configuration are integral. These properties hold for all Descartes configurations in the packing if they hold for one.

In this paper we study the geometric structure of super-packings, and their integrality properties. A striking geometric fact is that the circles in a super-packing do not overlap, as shown in §3. Figure 4 shows circles of curvature at most 65 in the super-packing generated from the $(0, 0, 1, 1)$ configuration in Figure 1 above. (The generating Descartes configuration is indicated with slightly darker lines.) Here the horizontal and vertical scales of the figure are roughly from $-2.2 \le x \le 2.2$ and $-2.2 \le y \le 2.2$. This picture is representative of the whole super-packing; we show in §6 that this super-packing is periodic under the two-dimensional lattice generated by the shifts $x \to x + 2$ and $y \to y + 2$.

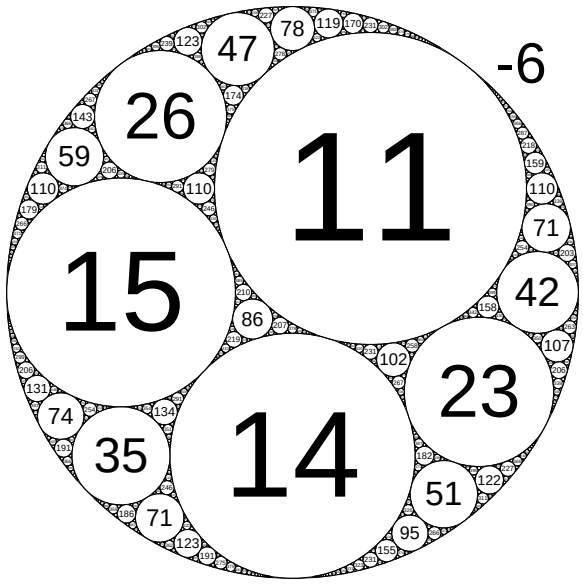

Figure 3: The integer Apollonian packing $(-6, 10, 11, 14)$

We cannot picture the whole super-packing because the circles in it are dense in the plane; they nest inside each other to an arbitrary depth. One can show that the set of points lying in the interior of an infinite nested sequence of such circles has full Lebesgue measure. However there is interesting structure visible if we turn up the scale of the circles. In §6 in Figure 8 we show all circles of curvature at most 200 inside the super-packing above, inside the unit square $0 \le x \le 1,\ 0 \le y \le 1$.

This paper is concerned with integrality properties of some super-packings, and we study successively stronger integrality properties. In §4 we require the curvatures to be integral, in §5 and §6 both the curvatures and curvature$\times$center must be integral, and in §8 the augmented curvature-center coordinates of all Descartes configurations in the super-packing must be integral. At this last level we show that there are exactly 14 such super-packings, viewed as rigid geometric objects. In §2 we give a more detailed description of our results.

The general framework of this paper was developed by the second author (JCL), who also did much of the writing. This paper is an extensive revision of a preprint written in 2000, and adds new results in §3 and §6.

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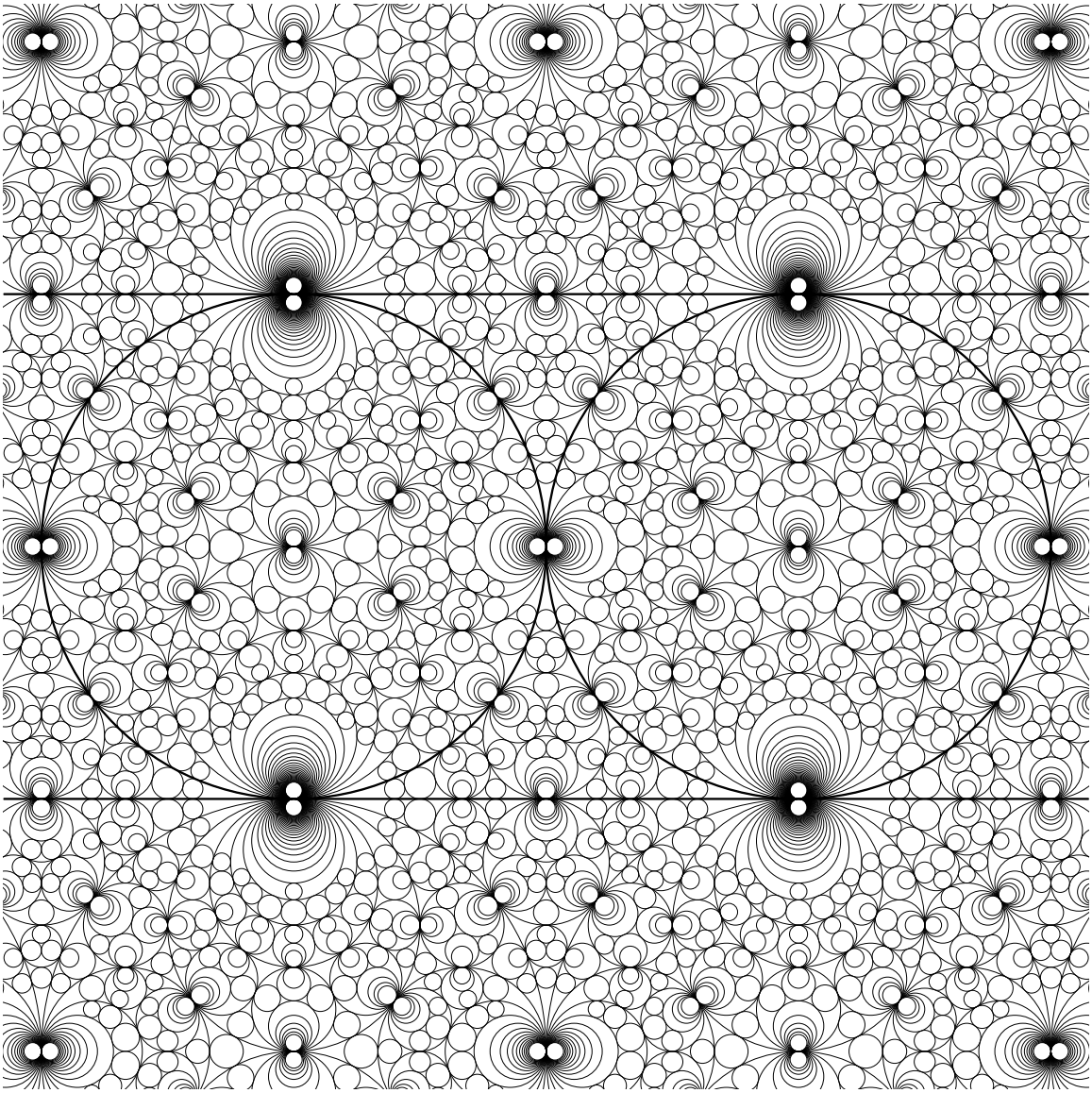

Figure 4: An initial part of the $(0, 0, 1, 1)$ super-packing (square of sidelength 4.4)

2. Summary of Main Results

We consider Apollonian super-packings. Analogously to part I, an Apollonian super-packing may be considered as either an geometric object or an algebraic object, as follows.

(i) [Geometric] A *geometric Apollonian super-packing* is a point set on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$, consisting of all the circles in four orbits of a certain group $G_{\mathcal{A}^S}(\mathcal{D})$ of Möbius transformations inside the conformal group Möb(2) acting on the four circles $\{C_1, C_2, C_3, C_4\}$ in a given Descartes configuration $\mathcal{D}$. The group $G_{\mathcal{A}^S}(\mathcal{D})$ depends on $\mathcal{D}$.

(ii) [Algebraic] An *(algebraic) Apollonian super-packing* is a set of ordered, oriented Descartes configurations, given by 48 orbits of the super-Apollonian group $\mathcal{A}^S[\mathcal{D}]$. The augmented curvature-center coordinates of its elements are $\mathcal{A}^S\mathbf{W}_{\mathcal{D}} := \{\mathbf{U}W_{\mathcal{D}} : \mathbf{U} \in \mathcal{A}^S\}$.

A geometric super-packing can be described in terms of its unordered, unoriented Descartes configurations. From this viewpoint, each geometric Apollonian super-packing corresponds to 48 different algebraic super-packings; there are 24 choices of ordering the four circles and 2 choices of total orientation of the configuration. We can consider geometric super-packings as unions of a countable number of Apollonian packings. This leads to interesting questions concerning the way these Apollonian packings are embedded inside the geometric super-packing. We note that as a point set, the geometric super-packing is invariant under the group action $G_{\mathcal{A}^S}(\mathcal{D})$. However it is not a closed set, and its closure is the entire Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$.

A large part of the paper considers integrality properties of curvatures and centers of some super-packings. These questions are mainly studied using algebraic super-packings, although we also consider questions concerning the associated geometric super-packing, such as its group of symmetries under Euclidean motions.

In §3 Theorem 3.1 shows that each geometric super-packing is a packing in the sense that the circles in it do not cross each other transversally, as mentioned above. Theorem 3.2 specifies certain subcollections of geometric super-packings which are genuine packings in the sense that the interiors of the circles do not overlap.

In §4 we study integer super-packings, in which all circles have integer curvatures. Integer super-packings are classified up to Euclidean motions by a single invariant, their *divisor* g, which is the greatest common divisor of the curvatures in any Descartes configuration in the packing. Theorem 4.1 shows that for each $g \geq 1$ there is a unique such integral super-packing, up to a Euclidean motion. We also show in Theorem 4.3 that for each geometric Apollonian circle packing that is integral, there exists a Euclidean motion taking it to one that is strongly integral.

In §5 we study strongly integral super-packings, which are those whose curvatures are integral and whose curvature×center is also integral. Strongly-integral super-packings are geometrically rigid: Theorem 5.1 shows that for each integer $g \geq 1$ there are exactly 8 strongly integral geometric super-packings which have divisor g. Here we do not allow the packings to be moved by Euclidean motions.

In §6 we study the relations between integer Apollonian packings and strongly integral super-packings. Without loss of generality we restrict to primitive integer super-packings, those with divisor 1. Theorem 6.1 shows that each of the 8 kinds of these has a large group of

internal symmetries, forming a crystallographic group. For convenience we fix one of them and call it the *standard strongly integral super-packing*; results proved for it have analogues for the other seven. Theorem 6.2 shows that each primitive integral Descartes configuration (except for the $(0,0,1,1)$ configuration) occurs in this packing with the center of its largest circle being contained in the closed unit square $0 \le x \le 1,\ 0 \le y \le 1$, and the location of this circle center is unique. Theorem 6.3 deduces that the geometric standard strongly integral super-packing contains a unique copy of each primitive integral Apollonian packing, except for the $(0,0,1,1)$ packing, having the property that the center of its bounding circle lies inside the closed unit square. Figures 5, 6 and 7 picture the locations of all primitive integer Apollonian packing with bounding circle of curvatures 6, 8 and 9, respectively. The unit square is indicated by slightly darker shading in the figures. Note that in Figure 5 the three Apollonian packings are generated by Descartes configurations with curvature vectors $(-6,7,42,43)$, $(-6,10,15,19)$ and $(-6,11,14,15)$; these are root quadruples in the sense of [5, Sec. 4].

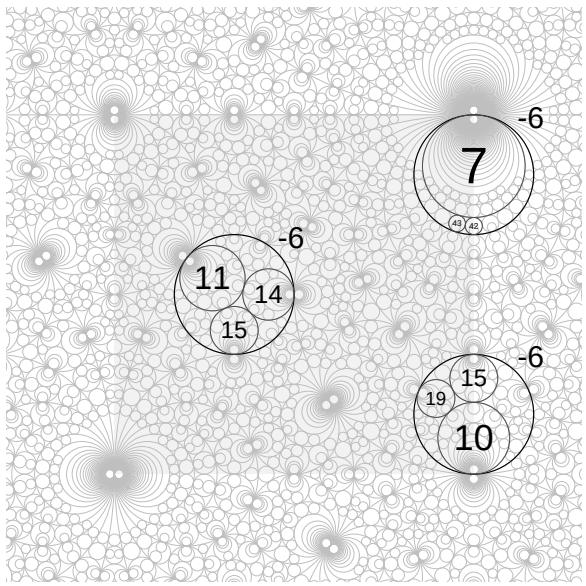

Figure 5: Integer Apollonian packings with bounding circle of curvature 6.

In §7 Theorem 7.1 shows that the super-Apollonian group is a finite index normal subgroup of the group $Aut(Q_D, \mathbb{Z})$ of integral automorphs of the Descartes quadratic form. The latter group can be identified with an index 2 subgroup of the integer Lorentz group $O(3,1,\mathbb{Z})$, and this identification allows us to identify the super-Apollonian group with a particular normal subgroup $\tilde{\mathcal{A}}^S$ of index 96 in $O(3,1,\mathbb{Z})$, defined after Theorem 7.1.

In §8 we study super-packings all of whose Descartes configurations are *super-integral* in the sense that their augmented curvature-center coordinate matrices $\mathbf{W}_{\mathcal{D}}$ are integer matrices. Theorem 8.1 shows there are exactly 14 geometric super-packings of this kind.

The last section §9 makes a few concluding remarks.

The Appendix considers minimal conditions to guarantee that a Descartes configuration is

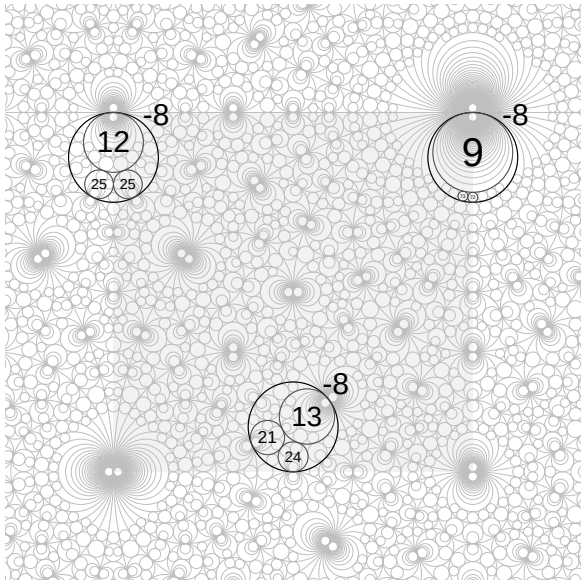

Figure 6: Integer Apollonian packings with bounding circle of curvature 8.

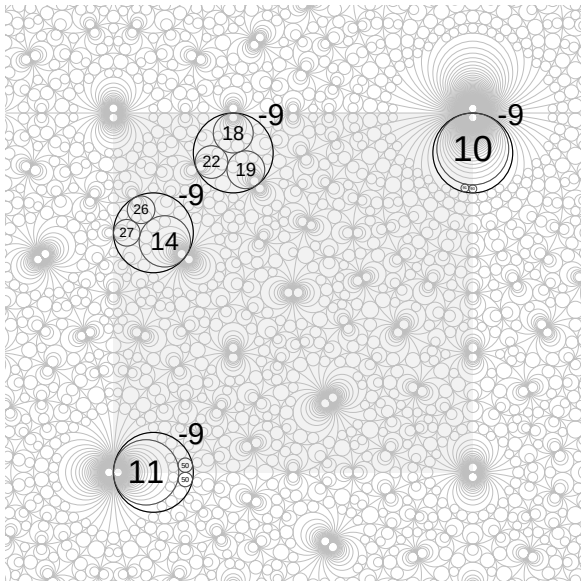

Figure 7: Integer Apollonian packings with bounding circle of curvature 9.

strongly integral. Theorem 10.1 shows that a configuration is strongly integral if (and only if) three of its four circles are strongly integral.

3. Geometric Apollonian Super-Packings

In this section we consider properties of geometric Apollonian circle packings. We view such a super-packing as a point set on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We first note that it is not a closed set. It is not hard to show that its closure is the whole Riemann sphere, Each geometric Apollonian packing has a group invariance property under a certain group of Möbius transformations which depends on the super-packing, which is the group generated by the countable set of groups of Möbius transformations that leave invariant some Apollonian packing contained in the super-packing.

Our object is prove the following "packing" property of geometric super-packings.

Theorem 3.1. *A geometric Apollonian super-packing is a circle packing in the weak sense that no two circles belonging to it cross each other transversally. Circles in the geometric super-packing may be nested, or tangent to each other.*

Before giving the proof, we describe the nature of the geometric packing in terms of nesting of circles. We view the packing $\mathcal{A}^S[\mathcal{D}_0]$ as generated from an initial (positively oriented) Descartes configuration $\mathcal{D}_0$, by multiplication by a finite set of generators of the super-Apollonian group. Each circle in the super-packing has a well-defined *nesting depth* d (relative to the generating configuration $\mathcal{D}_0$) which counts the number of circles in the packing which include C in their interior. Here the notion of "interior" is defined with respect to the initial Descartes configuration $\mathcal{D}_0$. The Apollonian group generators move "horizontally", leaving constant the nesting depth of any circles they produce. The dual Apollonian group generators move "vertically", by reflecting three of the circles in a configuration into the interior of the fourth circle, they increase the nesting depth by one. We show there is a unique "normal form" word of minimal length in the generators that produces a Descartes configuration $\mathcal{D}$ containing C. The nesting depth of C is exactly equal to the number d of generators of $\mathcal{A}^{\perp}$ that appear in this normal form word. The circles at nesting depth 0 are those circles in the Apollonian packing generated by $\mathcal{D}_0$. Each circle C at nesting depth $k \geq 1$ contains a unique Apollonian packing, consisting of it plus all circles at depth $k+1$ nested inside it.

Proof of Theorem 3.1. We view the geometric super-packing on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, so the initial Descartes configuration $\mathcal{D}_0$ consists of four circles on the sphere. In this case each circle defines a spherical cap. We choose an ordering and orientation of $\mathcal{D}_0$ (this does not affect the geometry), requiring that $\mathcal{D}_0$ have positive (total) orientation.

Let the super-packing be $\mathcal{A}^S[\mathcal{D}_0] = \mathcal{A}^S \mathbf{W}_{\mathcal{D}_0}$. We consider the effect of the super-Apollonian group generators acting on the left on the matrix $\mathbf{W}_{\mathcal{D}_0}$. The Descartes configurations in the super-packing are given by $\mathcal{D} = \mathbf{U}_m \mathbf{U}_{m-1} \cdots \mathbf{U}_1[\mathcal{D}_0]$ with each $\mathbf{U}_k \in \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_1^{\perp}, \mathbf{S}_2^{\perp}, \mathbf{S}_3^{\perp}, \mathbf{S}_4^{\perp}\}$. We consider words in the group measured by their *length* m. The stage m circles will consist of all new circles added using products of m generators. We may without loss of generality restrict to *normal form* words, which are those that satisfy the two conditions:

(i) If $\mathbf{U}_k = \mathbf{S}_i$, then $\mathbf{U}_{k+1} \neq \mathbf{S}_i$ and $\mathbf{U}_{k+1} \neq \mathbf{S}_j^{\perp}$ with $j \neq i$.

(ii) If $\mathbf{U}_k = \mathbf{S}_i^{\perp}$, then $\mathbf{U}_{k+1} \neq \mathbf{S}_i^{\perp}$.

Equivalently, looking backwards, if $\mathbf{U}_{k+1} = \mathbf{S}_i^{\perp}$ then either $\mathbf{U}_k = \mathbf{S}_i$ or else $\mathbf{U}_k = \mathbf{S}_j^{\perp}$ for some $j \neq i$. A word may be put in normal form by canceling adjacent equal generators, since all $\mathbf{S}_i^2 = (\mathbf{S}_i^{\perp})^2 = \mathbf{I}$, and by moving towards the right[2] in the word as far as possible any generator $\mathbf{S}_i^{\perp}$, using the property that it commutes with all $\mathbf{S}_j$ with $j \neq i$. These operations eventually put a word in normal form, without increasing its length. The operations do not change the Descartes configuration $\mathcal{D}$ it represents.

We prove the theorem by induction on the number of symbols m in a normal form word, which we call the stage of the induction. The induction hypotheses at stage m are as follows. Here we let C_i refer to the circle at row i of the associated (ordered, oriented) Descartes configuration.

(1) Each normal form word of length m produces either one or three new circles, according as $\mathbf{U}_m = \mathbf{S}_i$, where it is the circle C_i or $\mathbf{U}_m = \mathbf{S}_j^{\perp}$, where it is the three circles C_i with $i \neq j$.

(2) The nesting depth of any new circle produced at this stage is equal to the number of occurrences of a letter in $\mathcal{A}^{\perp}$ in its generating normal form word.

(3) Each new circle produced has empty interior, when it first appears.

In particular hypothesis (3) implies that all circles produced at stage m have disjoint interiors, and that each such circle contains no circles from earlier stages in its interior. If the induction is proved, then hypothesis (3) guarantees that the nesting depth of a circle is well-defined when it is first produced, because no new circle will ever include it in its interior. Hypothesis (3) also guarantees that all circles produced by the end of stage $m+1$ do not cross. As a consequence no two circles in the packing cross, using property (3) applied at that level m which is the greater of the levels of the two circles. Thus the theorem will follow.

The base case $m = 0$ of the induction is immediate, consisting of the initial Descartes configuration $\mathcal{D}_0$. We now show the induction step for $m + 1$, given m. We are given a Descartes configuration $\mathcal{D}' = \mathbf{U}_{m+1}[\mathcal{D}] = \mathbf{U}_{m+1}\mathbf{U}_m \cdots \mathbf{U}_1[\mathcal{D}_0]$ with a normal form word.

To establish hypothesis (1) for $m + 1$, suppose first that $\mathbf{U}_{m+1} = \mathbf{S}_i^{\perp}$. We assert that the i-th circle C_i of $\mathcal{D}$ was a new circle produced at stage m. For either $\mathbf{U}_m = \mathbf{S}_i$, in which case it was the unique new circle in $\mathcal{D}$ by induction hypothesis (1) at stage m, or $\mathbf{U}_m = \mathbf{S}_j^{\perp}$ with $j \neq i$, in which case it was one of three new circles produced at stage m. By induction hypothesis (2) the circle C_i has empty interior at stage m. The three circles $\{C_j' : j \neq i\}$ in the new configuration $\mathcal{D}'$ are contained in the interior of C_i are therefore new circles. They do not cross, being part of a Descartes configuration. Thus hypothesis (1) holds for $m+1$ in this case.

Suppose next that $\mathbf{U}_{m+1} = \mathbf{S}_i$, so the possible new circle is C_i'. If $\mathcal{D}_k := \mathbf{U}_k\mathbf{U}_{k-1} \cdots \mathbf{U}_1[\mathcal{D}_0]$ is the maximal length subword such that $\mathbf{U}_k = \mathbf{S}_j^{\perp}$ for some j, with $1 \leq k \leq m$ then $\mathcal{D}'$ belongs to the Apollonian packing generated by $\mathcal{D}_k$, since all subsequent generators belong to the Apollonian group. If no such k exists, then $\mathcal{D}'$ is in the Apollonian packing generated by the original Descartes configuration $\mathcal{D}_0$. For $k \geq 1$, this Apollonian packing is entirely contained in the interior of a bounding circle $C = C_j$ first produced at stage $k - 1$. At that time the interior of C was empty, by induction hypotheses (2) and (3) at stage $k-1$. The only Descartes configurations that ever can enter the interior of the circle C, must do so by a reflection in C and these are exactly those normal form words starting with initial segment $\mathbf{U}_k\mathbf{U}_{k-1} \cdots \mathbf{U}_1$.

[2]That is, moving it towards the beginning of the word.

(This follows from uniqueness of a circle when it is created, hypothesis (3) applied at stage k.) The words contain this initial segment at the same depth, with all subsequent letters in $\mathcal{A}$, fill out an Apollonian packing at this depth. In particular each such normal form word produces one new circle in this Apollonian packing. Recall that all the circles in the Apollonian packing inside the bounding circle C have disjoint interiors (Theorem 4.1 of part I). These circles all have the same depth, and (1) holds in this case. Normal form words that have another subsequent generator in $\mathcal{A}^{\perp}$ confine the resulting Descartes configuration to the inside of a single circle in the Apollonian packing $\mathcal{A}[\mathcal{D}_k]$ already produced, and all longer words with this prefix are entirely contained inside this circle. In particular, they do not coincide with $C_i^{'}$. the new circle produced by the normal form word corresponding to $\mathcal{D}'$. It follows that the circle $C_i^{'}$ is new, so hypothesis (1) holds for $m+1$ in this case.

There remains the case where $\mathbf{U}_{m+1} = \mathbf{S}_i$ and all previous $\mathbf{U}_k \in \mathcal{A}$. Then $\mathbf{U}_{m+1} \cdots \mathbf{U}_1[\mathcal{D}_0]$ belongs to the Apollonian packing generated by $\mathcal{D}_0$, and is at depth 0. Here $C_i^{'}$ is new since the generation of an Apollonian packing creates one new circle at each step, and any word that contains an element of $\mathcal{A}^{\perp}$ moves the Descartes configuration inside an older circle in this packing, from which it cannot escape. Thus $C_i^{'}$ is a new circle in this case, and hypothesis (1) holds.

Hypothesis (2) holds for $m+1$ in the case $\mathbf{U}_m = \mathbf{S}_i^{\perp}$ because the three new circles produced have nesting depth one greater than the nesting depth of C_i at the previous level, to which induction hypothesis (2) applied. It also holds in the remaining case $\mathbf{U}_m = \mathbf{S}_i$ because the argument above showed that the nesting depth did not increase in this case.

Hypothesis (3) holds if $\mathbf{U}_{m+1} = \mathbf{S}_i^{\perp}$. Now circle C_i was first created only at stage m, and the only other possible sequences leading to a Descartes configuration including this circle must start from $\mathcal{D}_m$ and use a generator $\mathbf{U}_{m+1} = \mathbf{S}_j$ with $j \neq i$. In all other cases the resulting Descartes configuration includes no circle inside C_i, so the interiors of the three new circles produced are empty at the end of stage $m+1$.

In the remaining case $\mathbf{U}_{m+1} = \mathbf{S}_i$, we have already observed that the new circle $C_i^{'}$ produced, is disjoint from all other circles in the Apollonian packing $\mathcal{A}[\mathcal{D}_k]$ created by level m, which are necessarily contained in the bounding circle C of $\mathcal{D}_k$. As mentioned above, all Descartes configurations containing a circle inside C must have an initial segment of their generating word, giving the unique normal form word that first generates C, at stage $k \leq m$. Other depth $m+1$ words with this initial segment, and with all subsequent $\mathbf{U}_j$ drawn from the Apollonian group generators produce new circles in the Apollonian packing $\mathcal{A}[\mathcal{D}_k]$, disjoint from $C_i^{'}$. Any normal form word at depth $m+1$ with this initial segment, which contain some generator $\mathbf{S}_j^{\perp}$ after-wards, produce a Descartes configuration contained inside a circle of the Apollonian packing $\mathcal{A}[\mathcal{D}_k]$ different from C. It follows that the interior of $C_i^{'}$ is empty at the end of stage $m+1$, as required.

This completes the induction step and the proof. ∎

Super-packings have some properties that are genuine packing properties. The proof of Theorem 3.1 established in hypothesis (3) shows that the finite set of circles at stage m of the construction, starting from a generating Descartes configuration $\mathcal{D}$, all had disjoint interiors. It also gives the following stronger result.

Theorem 3.2. *For a geometric Apollonian super-packing given with a generating Descartes*

configuration $\mathcal{D}$, and each $k \geq 1$, the set of all circles having nesting depth exactly k with respect to $\mathcal{D}$ have pairwise disjoint interiors. These circles can be viewed as forming an infinite collection of Apollonian packings, each missing one circle; the missing circle is a bounding circle at depth $k-1$.

Proof. The proof of Theorem 3.1 shows that the nesting depth of a circle is well-defined. Circles at nesting depth k have disjoint interiors, since no two of them cross, and the only way for two of them to have an interior point in common is for one to be nested inside the other, which would violate the nesting ordering.

Given a normal form word $\mathbf{U} := \mathbf{U}_m \mathbf{U}_{m-1} \cdots \mathbf{U}_1$ with $\mathbf{U}_m = \mathbf{S}_i^{\perp}$, and containing exactly k elements of $\mathcal{A}^{\perp} = \langle \mathbf{S}_1^{\perp}, \mathbf{S}_2^{\perp}, \mathbf{S}_3^{\perp}, \mathbf{S}_4^{\perp} \rangle$, the set of all normal form words having $\mathbf{V}$ as prefix and all other letters $\mathbf{U}_j \in \mathcal{A}$ for $j > m$ produce all the circles in an Apollonian packing $\mathcal{A}[\mathcal{D}_m]$, with $\mathcal{D}_m = \mathbf{U}[\mathcal{D}_0]$. All these circles are at depth k except for the outer circle of $\mathcal{D}_m$, which is its i-th circle. Enumerating all possible such $\mathbf{U}$ as prefixes represents the set of nesting depth k circles as a collection of Apollonian packings, each excluding one circle when $k \geq 1$. ∎

Remarks. (1) The proof of Theorem 3.1 is a geometric analogue of the presentation for the super-Apollonian group proved in part I, [3, Theorem 6.1].

(2) The analogous result to Theorem 3.1 fails to hold in all dimensions $n \geq 4$, as explained in part III. The "nesting" property of the dual Apollonian configurations resulting from "vertical" moves still exists and works in all dimensions. However the "horizontal" motions moving spheres around in Apollonian packings produces spheres that cross in dimensions $n \geq 4$, see [4, Lemma 4.1].

(3) We cannot easily visualize a geometric super-packing as a completed object because the circles in it are dense in the plane. We can however picture a partial version of it that pictures all circles of size above a given threshold, in some finite region of the packing. The integral super-packings we are most interested in have a periodic lattice of symmetries (see Theorem 6.1), so it suffices to examine a finite region of the packing. Figure 4 in §1 and Figure 8 in §6 exhibit part of a super-packing.

(4) Every circle C in a geometric super-packing $\mathcal{A}^S[\mathcal{D}]$ has associated to it a unique Apollonian packing of which it is the bounding circle. If it is a depth k circle (relative to the starting configuration $\mathcal{D}$), then this Apollonian packing consists of all depth $k+1$ circles contained in the interior of C.

4. Integral Super-Packings

An Apollonian super-packing is *integral* if it contains one (and hence all) Descartes configuration whose circles have integer curvatures.

An invariant of an integral super-packing is its *divisor* g, which is the greatest common divisor of the curvatures of the circles in any Descartes configuration in the super-packing. The quantity g is well-defined independent of the Descartes configuration chosen, using the relation $\mathbf{M}_{\mathcal{D}'} = \mathbf{U}\mathbf{M}_{\mathcal{D}}$ between two such configurations, where $\mathbf{U} \in \mathcal{A}^S$. Since $\mathbf{U}$ is an integer

matrix with determinant ± 1, it preserves the greatest common divisor of each column of the integer matrix $\mathbf{M}_{\mathcal{D}}$. Here the first column encodes the curvatures of the circle.

Theorem 4.1. *For each integer $g \geq 1$ there exists an integral Apollonian super-packing with divisor g. The associated geometric super-packing is unique, up to a Euclidean motion.*

As an immediate corollary of this result, the integral super-packing with divisor g contains at least one copy of every integral Descartes configuration having divisor g. For each such a Descartes configuration generates an integral super-packing with divisor g, so the corollary follows by the uniqueness assertion.

We defer the proof of Theorem 4.1 to the end of the section. It is based on a reduction theory which finds inside any such integral super-packing a Descartes configuration having particularly simple curvatures.

Theorem 4.2. *Let $\mathcal{D}$ be an integral Descartes configuration, with divisor $g := \gcd(b_1, b_2, b_3, b_4)$. Then the Apollonian super-packing $\mathcal{A}^S[\mathcal{D}]$ generated by $\mathcal{D}$ contains a Descartes configuration having curvature vector a permutation of either $(0, 0, g, g)$ or $(0, 0, -g, -g)$, with the former case occurring if $b_1 + b_2 + b_3 + b_4 > 0$ and the latter case if $b_1 + b_2 + b_3 + b_4 < 0$.*

Proof. Since the super-Apollonian group $\mathcal{A}^S$ preserves the (total) orientation of Descartes configurations, it is sufficient to show that for a positively oriented integral Descartes configuration $\mathcal{D}$ with curvatures (b_1, b_2, b_3, b_4), $b_1 + b_2 + b_3 + b_4 > 0$, there exists $\mathbf{U} \in \mathcal{A}^S$ and a permutation matrix $\mathbf{P}_\sigma$ such that

$$\mathbf{P}_\sigma \mathbf{U}(b_1, b_2, b_3, b_4)^T = (0, 0, g, g)^T.$$

We measure the *size* of the curvature vector $\mathbf{v} = (b_1, b_2, b_3, b_4)^T$ of a Descartes configuration by

$$size(\mathbf{v}) := \mathbf{1}^T \mathbf{v} = b_1 + b_2 + b_3 + b_4.$$

We claim that for positively oriented integral Descartes configurations with greatest common divisor g, we have

$$size(\mathbf{v}) \geq 2g,$$

and equality holds if and only if $\mathbf{v}$ is a permutation of $(0, 0, g, g)$. If all curvatures are nonnegative this is clear, since at most two can be zero, and the other two are positive integers. Now in any Descartes configuration at most one circle can have negative curvature, call it $b_1 = -a$ ($a \in \mathbb{Z}_+$), in which case it encloses the other three. Each of these three enclosed circles has a larger curvature in absolute value than the bounding circle, so $b_i \geq a + 1$ for $i = 2, 3, 4$. Thus $size(\mathbf{v}) \geq -a + 3(a + 1) \geq 2a + 3 > 2g$, which proves the claim.

We give a reduction procedure which chooses matrices in $\mathcal{A}^S$ to reduce the size and show that the procedure halts only at a vector of form $(0, 0, g, g)$, up to a permutation. To specify it, we observe that for the curvature vector $\mathbf{v} = (b_1, b_2, b_3, b_4)^T$ of any integral Descartes configuration with $b_1 \leq b_2 \leq b_3 \leq b_4$, we have

$$size(\mathbf{S}_4 \mathbf{v}) = \mathbf{1}^T \mathbf{S}_4 \mathbf{v} \leq \mathbf{1}^T \mathbf{v} = size(\mathbf{v}). \tag{4.1}$$

and equality holds if and only if $b_1b_2+b_2b_3+b_3b_1=0$. To see this, we have $\mathbf{S}_4\mathbf{v}=(b_1,b_2,b_3,b_4')^T$ where

$$b_4'=2(b_1+b_2+b_3)-b_4=b_1+b_2+b_3-2\sqrt{b_1b_2+b_2b_3+b_3b_1}.$$

Thus

$$\mathbf{1}^T\mathbf{S}_4\mathbf{v}-\mathbf{1}^T\mathbf{v}=b_4'-b_4=-4\sqrt{b_1b_2+b_2b_3+b_3b_1}\leq 0.$$

Equality can hold if and only if $b_1b_2+b_2b_3+b_3b_1=0$, which proves the observation. Note that $b_1\leq 0$ when the equality in (4.1) holds. Also note that $g=\gcd(b_1,b_2,b_3,b_4)$ is an invariant under the action of $\mathcal{A}^S$.

Starting with any positively oriented integral Descartes configuration with curvature vector $(b_1,b_2,b_3,b_4)^T$, where b_i is the largest number, we apply $\mathbf{S}_i\in\mathcal{A}$. By (4.1), the $size(\mathbf{v})$ decreases but cannot be negative, so after a finite series of $\mathbf{S}_i$ we arrive at a positively oriented integral Descartes configuration with curvatures $\mathbf{v}'=(b_1',b_2',b_3',b_4')^T$, where $\gcd(b_1',b_2',b_3',b_4')=g$ and the smallest curvature, say b_1', satisfies $b_1'\leq 0$, and the size of $\mathbf{v}'$ can not be reduced by the action of the Apollonian group $\mathcal{A}$. Call this the basic reduction step. Note that the basic reduction step involves only matrices in the Apollonian group and therefore moves around inside a single Apollonian packing.

If $b_1'=0$ then necessarily $b_2'=0$, whence the curvature vector is $(0,0,b_3',b_3')^T$, and by $g=\gcd(b_1',b_2',b_3',b_4')$, we have $b_3'=g$ and the reduction halts. If $b_1'<0$, applying $\mathbf{S}_1^T$, we get a new Descartes configuration with $\mathbf{v}''=(-b_1',b_2'+2b_1',b_3'+2b_1',b_4'+2b_1')^T$, which is positively oriented and lies in a new Apollonian packing and has

$$size(\mathbf{v}'')=size(\mathbf{v})+4b_1'<size(\mathbf{v}).$$

Thus the size strictly decreases and is non-negative. Now we may re-apply the basic reduction step. Continuing in this way we get strict decrease of $size(\mathbf{v})$ at each step, with the only possible halting step being the smallest curvature equals 0. Since the size of the curvature vector is bounded below and decreases by at least one at each step, the procedure terminates at $(0,0,g,g)$, up to a permutation. ∎

Proof of Theorem 4.1. For existence, the super-packing generated by a Descartes configuration with curvature vector $(0,0,g,g)$, which is a homothetically scaled version of the configuration $(0,0,1,1)$ pictured in Figure 1, is necessarily integral with divisor g.

For uniqueness, Theorem 4.2 shows that any two geometric integral Apollonian super-packings with divisor g each contain a Descartes configuration whose curvatures are $(0,0,g,g)$ up to permutation and orientation. Now it is true for any two such Descartes configurations with identical curvature vectors are congruent, i.e. one is obtainable from the other by a Euclidean motion. This is obvious by inspection for the $(0,0,g,g)$ Descartes configuration, which necessarily consists of two touching circles of radius $\frac{1}{g}$ and two parallel lines.

Now the Euclidean motion that takes one Descartes configuration to the other, also takes the super-packing generated by the first configuration to the one generated by the other, because the super-packing is defined by the action of the super-Apollonian group on the left on the Descartes configuration $\mathbf{W}_{\mathcal{D}}$, and this commutes with the Euclidean motion acting as a Möbius transformation on the right. This establishes uniqueness. ∎

We can use the freedom of a Euclidean motion allowed in Theorem 4.1 to make an internal Apollonian super-packing strongly integral.

Theorem 4.3. *For each integral geometric Apollonian super-packing there is a Euclidean motion that takes it to a strongly integral geometric Apollonian super-packing.*

Proof. Using Theorem 4.2 each integral geometric Apollonian packing contains a Descartes configuration $\mathcal{D}$ with curvatures $(0, 0, g, g)$; note that for a geometric packing the order and orientation of the Descartes configuration do not matter. The curvature vector determines the Descartes configuration up to congruence. We can now find a strongly integral Descartes configuration $\mathcal{D}'$ with this curvature vector. For $g = 1$ such a configuration is given explicitly by (6.1) below, and for larger g we obtain a strongly integral configuration from it using the homothety $(x, y) \mapsto \frac{1}{g}(x, y)$. There exists a Euclidean motion that maps $\mathcal{D}$ to $\mathcal{D}'$, since they are congruent configurations. This motion maps the super-packing $\mathcal{A}^S[\mathcal{D}]$ to the super-packing $\mathcal{A}^S[\mathcal{D}']$, which is strongly integral. ∎

5. Strongly Integral Super-packings

A Descartes configuration $\mathcal{D}$ is *strongly integral* if its associated 4×3 curvature center-coordinate matrix $\mathbf{M}_{\mathcal{D}}$ is an integer matrix; this property is independent of ordering or orientation of the Descartes configuration. A super-packing is called *strongly integral* if it contains one (and hence all) Descartes configurations having this property. Since a strongly integral super-packing is integral, it has a divisor g as an invariant.

Our main object in this section is to classify strongly integral super-packings, as follows.

Theorem 5.1. *(1) For each $g \geq 1$ there are exactly 8 different geometric Apollonian super-packings that are strongly integral and have divisor g.*

(2) The set of all ordered, oriented Descartes configurations that are strongly integral and have a given divisor g fill exactly 384 orbits of the super-Apollonian group.

This theorem classifies these super-packings as rigid objects, not allowed to be moved by Euclidean motions. To prove this result we derive a normal form for a "super-root quadruple" in a super-packing of the kind above, as follows.

Theorem 5.2. *Given a strongly integral Apollonian super-packing $\mathcal{A}^s[\mathcal{D}_0]$ with the divisor $g \geq 1$, there exists a unique "reduced" Descartes configuration $\mathcal{D} \in \mathcal{A}^S[\mathcal{D}_0]$ whose curvature-center coordinate matrix $\mathbf{M} = \mathbf{M}_{\mathcal{D}}$ is of the form $\mathbf{A}_{m,n}[g]$ or $\mathbf{B}_{m,n}[g]$ for $m, n \in \{0, 1\}$, up to a permutation of rows, where*

$$\mathbf{A}_{m,n}[g] = \pm \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ g & m & n \\ g & m-2 & n \end{bmatrix} \qquad \mathbf{B}_{m,n}[g] = \pm \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ g & m & n \\ g & m & n-2 \end{bmatrix}, \tag{5.1}$$

and the sign is determined by the orientation of $\mathcal{D}_0$.

Proof. For a strongly integral Apollonian super-packing $\mathcal{A}^S[\mathcal{D}_0]$ with the divisor g, by Theorem 4.2 there exists a strongly integral Descartes configuration $\mathcal{D} \in \mathcal{A}_S[\mathcal{D}_0]$ with curvatures $\pm(0, 0, g, g)$. The two straight lines in $\mathcal{D}$ must be parallel to either x-axis or y-axis. It follows

that the 4×3 curvature-center coordinate matrix $\mathbf{M}_{\mathcal{D}}$ is of the form $\mathbf{A}_{m,n}[g]$, or $\mathbf{B}_{m,n}[g]$, for some $m, n \in \mathbb{Z}$, up to a permutation of rows.

We now reduce m, n to take the values $0, 1$ using the following identities, which are easy to check.

$$\begin{aligned}
\mathbf{P}_{(34)}\mathbf{S}_3\mathbf{A}_{m,n}[g] &= \mathbf{A}_{m-2,n}[g], & \mathbf{P}_{(34)}\mathbf{S}_3\mathbf{B}_{m,n}[g] &= \mathbf{B}_{m,n-2}[g],\\
\mathbf{P}_{(34)}\mathbf{S}_4\mathbf{A}_{m,n}[g] &= \mathbf{A}_{m+2,n}[g], & \mathbf{P}_{(34)}\mathbf{S}_4\mathbf{B}_{m,n}[g] &= \mathbf{B}_{m,n+2}[g],\\
\mathbf{P}_{(12)}\mathbf{S}_1^T\mathbf{A}_{m,n}[g] &= \mathbf{A}_{m,n+2}[g], & \mathbf{P}_{(12)}\mathbf{S}_1^T\mathbf{B}_{m,n}[g] &= \mathbf{B}_{m+2,n}[g],\\
\mathbf{P}_{(12)}\mathbf{S}_2^T\mathbf{A}_{m,n}[g] &= \mathbf{A}_{m,n-2}[g], & \mathbf{P}_{(12)}\mathbf{S}_2^T\mathbf{B}_{m,n}[g] &= \mathbf{B}_{m-2,n}[g].
\end{aligned}$$

where $\mathbf{P}_{(ij)}$ is the permutation matrix that exchanges i and j. Also note that $\mathbf{P}_\sigma\mathbf{S}_i = \mathbf{S}_{\sigma(i)}\mathbf{P}_\sigma$, $\mathbf{P}_\sigma\mathbf{S}_i^T = \mathbf{S}_{\sigma(i)}^T\mathbf{P}_\sigma$. Hence there is a series of group operations in $\mathcal{A}^S$ which takes $\mathbf{M}_{\mathcal{D}}$ to a permutation of $\mathbf{A}_{m,n}[g]$ or $\mathbf{B}_{m,n}[g]$ with $m, n \in \{0, 1\}$. This proves the existence of the "reduced" Descartes configuration in the Apollonian super-packing $\mathcal{A}^S[\mathcal{D}_0]$.

To prove the uniqueness, it suffices to show that the $24 \times 8 \times 2 = 384$ Descartes configurations whose curvature-center coordinate matrices are

$$\{\mathbf{P}\mathbf{A}_{m,n}[g], \mathbf{P}\mathbf{B}_{m,n}[g] \mid \mathbf{P} \in \mathrm{Perm}_4, m, n \in \{0, 1\}\}$$

are in different Apollonian super-packings. (There are two signs for each of $\mathbf{A}_{m,n}[g]$ and $\mathbf{B}_{m,n}[g]$.) In what follows we let $\tilde{\mathbf{A}}_{m,n}[g]$, and $\tilde{\mathbf{B}}_{m,n}[g]$ denote the unique 4×4 augmented curvature-center coordinate matrices extending $\mathbf{A}_{m,n}[g]$ and $\mathbf{B}_{m,n}[g]$, respectively; uniqueness holds by Theorem 3.1 of part I.

Note that each $\mathbf{S}_i$ and $\mathbf{S}_i^T$ preserves the (total) orientation of the Descartes configuration, as well as the parity of every element of $\mathbf{M}_{\mathcal{D}}$. First we show that the matrices $\mathbf{A}_{m,n}[g], \mathbf{B}_{m,n}[g], (m, n \in \{0, 1\}$ are in distinct orbits of $\mathcal{A}^S \times \mathrm{Perm}_4$. To see this, for each integral vector $\mathbf{v} \in \mathbb{Z}^4$, let $\kappa(\mathbf{v})$ be the number of even terms in $\mathbf{v}$, and for any strongly integral Descartes configuration $\mathcal{D}$ let

$$\kappa(\mathbf{M}_{\mathcal{D}}) = (\kappa(\mathbf{v}_1), \kappa(\mathbf{v}_2), \kappa(\mathbf{v}_3)),$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the column vectors of $\mathbf{M}_{\mathcal{D}}$. Then $\kappa(\mathbf{M}_{\mathcal{D}})$ is invariant under the action of $\mathcal{A}^S \times \mathrm{Perm}_4$. For $m, n \in \{0, 1\}$, $\kappa(\mathbf{A}_{m,n}[g]), \kappa(\mathbf{B}_{m,n}[g])$ are all distinct except $\kappa(\mathbf{A}_{1,0}[g]) = \kappa(\mathbf{B}_{0,1}[g]) = (*, 2, 2)$, where $*$ is 2 if g is odd, and 4 if g is even. However, $\mathbf{A}_{1,0}[g]$ and $\mathbf{B}_{0,1}[g]$ can not be equivalent under the action of $\mathcal{A}^S \times \mathrm{Perm}_4$. Arguing by contradiction, assume that there exists a matrix $\mathbf{U} \in \mathcal{A}^S \times \mathrm{Perm}_4$ such that $\mathbf{U}\mathbf{A}_{1,0}[g] = \mathbf{B}_{0,1}[g]$. This relation lifts to augmented curvature-center coordinates: $\mathbf{U}\tilde{\mathbf{A}}_{1,0}[g] = \tilde{\mathbf{B}}_{0,1}[g]$. It follows that $\mathbf{U} = (\tilde{\mathbf{A}}_{1,0}[g])^{-1}\tilde{\mathbf{B}}_{0,1}[g]$ is unique. We can directly verify that for $m, n \in \mathbb{Z}$

$$\tilde{\mathbf{A}}_{m,n}[g] = \begin{bmatrix} 2(n+1)/g & 0 & 0 & 1 \\ 2(1-n)/g & 0 & 0 & -1 \\ (m^2+n^2-1)/g & g & m & n \\ ((m-2)^2+n^2-1)/g & g & m-2 & n \end{bmatrix},$$

$$\tilde{\mathbf{B}}_{m,n}[g] = \begin{bmatrix} 2(m+1)/g & 0 & 1 & 0 \\ 2(1-m)/g & 0 & -1 & 0 \\ (m^2+n^2-1)/g & g & m & n \\ (m^2+(n-2)^2-1)/g & g & m & n-2 \end{bmatrix}, \tag{5.2}$$

by checking that these satisfy the identity of Theorem 3.2 of part I necessary and sufficient to be augmented curvature-center coordinates. Now it is easy to verify that $\mathbf{P}_{(14)}\mathbf{P}_{(23)}\mathbf{D}\tilde{\mathbf{A}}_{1,0}[g] = \tilde{\mathbf{B}}_{0,1}[g]$, where $\mathbf{D} = -\mathbf{Q}_D$ is defined as in §3 of Part I [3],

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}. \tag{5.3}$$

By the uniqueness $\mathbf{P}_{(14)}\mathbf{P}_{(23)}\mathbf{D} = \mathbf{U} \in \mathcal{A}^S \times \mathrm{Perm}_4$, which is impossible since $\mathcal{A}^S \times \mathrm{Perm}_4$ consists of integral matrices only, while $\mathbf{D}$ has half integers. In conclusion, $\mathbf{A}_{m,n}[g], \mathbf{B}_{m,n}[g]$, $(m, n \in \{0, 1\})$ are in distinct orbits of $\mathcal{A}^S \times \mathrm{Perm}_4$.

The final step is to show that for any permutation $\mathbf{P} \neq \mathbf{I}$, $\mathbf{P}\mathbf{A}_{m,n}[g]$ (resp. $\mathbf{P}\mathbf{B}_{m,n}[g]$) can not be obtained from $\mathbf{A}_{m,n}[g]$ (resp. $\mathbf{B}_{m,n}[g]$) by an action of $\mathcal{A}^S$. That is, we claim: if for a permutation matrix $\mathbf{P} \in \mathrm{Perm}_4$, there exists a matrix $\mathbf{U} \in \mathcal{A}^S$ such that

$$\mathbf{U}\mathbf{A}_{m,n}[g] = \mathbf{P}\mathbf{A}_{m,n}[g], \qquad \text{or } \mathbf{U}\mathbf{B}_{m,n}[g] = \mathbf{P}\mathbf{B}_{m,n}[g],$$

then $\mathbf{P} = \mathbf{I}$.

To establish the claim, consider again the 4×4 ACC-coordinate matrices $\tilde{\mathbf{A}}_{m,n}[g]$ and $\tilde{\mathbf{B}}_{m,n}[g]$. From §3.1 of Part I [3], for any Descartes configuration $\mathcal{D}$, the curvature-center coordinate matrix $\mathbf{M}_{\mathcal{D}}$ can be uniquely extended to a 4×4 ACC-coordinate matrix $\mathbf{W}_{\mathcal{D}}$. It follows that if $\mathbf{U}\mathbf{A}_{m,n}[g] = \mathbf{P}\mathbf{A}_{m,n}[g]$, then the equality holds for their 4×4 ACC-coordinate matrices, i.e., $\mathbf{U}\tilde{\mathbf{A}}_{m,n}[g] = \mathbf{P}\tilde{\mathbf{A}}_{m,n}[g]$. It implies $\mathbf{U} = \mathbf{P} \in \mathcal{A}^S \cap \mathrm{Perm}_4$. However, comparing the size of $\mathbf{U}$ and $\mathbf{P}$, where the size of a matrix $\mathbf{U}$ is defined as $f(\mathbf{U}) := \mathbf{1}^T\mathbf{U}\mathbf{1}$, we have $f(\mathbf{P}) := \mathbf{1}^T\mathbf{P}\mathbf{1} = 4$ for $\mathbf{P} \in \mathrm{Perm}_4$, and $f(\mathbf{U}) := \mathbf{1}^T\mathbf{U}\mathbf{1} \geq 8$ for any $\mathbf{U} \in \mathcal{A}^S, \mathbf{U} \neq \mathbf{I}$, (c.f.§5 of Part I [3]). Therefore the only possibility is $\mathbf{U} = \mathbf{P} = \mathbf{I}$. The same argument applies to $\mathbf{B}_{m,n}[g]$, and the claim follows.

We conclude that a reduced Descartes configuration of the form (5.2) in any strongly integral Apollonian super-packing exists and is unique. ∎

Proof of Theorem 5.1. (1) Since there are 48 orbits of ordered, oriented Descartes configurations corresponding to each geometric super-packing, to show there are exactly 8 geometric super-packings it suffices to show that the strongly integral Descartes configurations form 384 orbits of the Apollonian group, which we do below.

(2) We enumerate the complete set of ordered, oriented Descartes configurations that are strongly integral, with greatest common divisor g, as follows. By Theorem 5.2, any such Descartes configuration is equivalent under the action of $\mathcal{A}^S$ to a permutation of a Descartes configuration whose 4×3 curvature-center coordinate matrix is of the form $A_{m,n}$ or $B_{m,n}$, with $m, n \in \{0, 1\}$. The uniqueness of Theorem 5.2 asserts that the 24 permutations of $A_{m,n}[g]$ ($B_{m,n}[g]$) are all in distinct orbits of the super-Apollonian group. Considering the two choices of orientations, we get $24 \times 8 \times 2 = 384$ orbits. ∎

6. Primitive Strongly Integral Super-packings

The strongly integral superpackings having a given curvature g.c.d. g are each obtainable from one with $g = 1$ by homothety. A homothety $(x, y) \mapsto r(x, y)$ changes all curvatures by $\frac{1}{r}$ while leaving (curvature)$\times$(center) unchanged. Applying the homothety with $r = \frac{1}{g}$ takes a strongly integral super-packing with g.c.d. g to one with g.c.d. equal to 1.

We now consider strongly integral super-packings having $g = 1$, which we call *primitive* super-packings. Results for them carry over easily to those with divisor $g > 1$ by applying a homothety. Theorem 5.1 showed there are exactly 8 such packings.

For convenience we will single out a particular one of them and term it the *standard strongly integral super-packing*. We choose this to be the super-packing generated by the ordered, oriented Descartes configuration having

$$\mathbf{M}_{\mathcal{D}_1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad \text{so that} \qquad \mathbf{W}_{\mathcal{D}_1} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}. \tag{6.1}$$

This corresponds to a $(0, 0, 1, 1)$ Descartes quadruple, with the centers of the two circles lying along the x-axis and the circles touching at the origin $(0, 0)$. The associated geometric integral super-packing is the one pictured in §3. Results proved below for the standard super-packing apply generally to all eight primitive integral super-packings, using the Euclidean motions mapping between them described after the proof of Theorem 6.1 below.

We first show that the geometric standard strongly-integral super-packing has a large group of symmetries, which form a crystallographic group of the plane.

Theorem 6.1. *The geometric standard strongly integral super-packing is invariant under the following Euclidean motions:*

(1) The lattice of translations $(x, y) \mapsto (x + 2, y)$, and $(x, y) \mapsto (x, y + 2)$,

(2) The reflections $(x, y) \mapsto (-x, y)$, and $(x, y) \mapsto (x, -y)$.

The crystallographic group generated by these motions is the complete set of Euclidean motions leaving the geometric standard strongly integral super-packing invariant.

Proof. The key fact used is that the action of the super-Apollonian group on Descartes configurations commutes with the action of Euclidean motions acting on Descartes configurations as Möbius transformations. This was shown in part I [3, Theorem 3.3(4)].

(1) There is a Descartes configuration corresponding to $\mathcal{D}_0$ shifted by 2 in the x-direction and the y-direction; call them $\mathcal{D}_0^x$ and $\mathcal{D}_0^y$, respectively. These are given by the actions of $\mathbf{S}_4$ and $\mathbf{S}_1^{\perp}$, respectively. Treating the x-shift first, we then have

$$\mathcal{A}^S[\mathcal{D}_0] = \mathcal{A}^S[\mathcal{D}_0^x] = \mathcal{A}^S[\mathfrak{t}_x(\mathcal{D}_0^{'})],$$

in which $\mathfrak{t}_x : \mathbf{z} \mapsto \mathbf{z} + 2$ is the Euclidean motion translation by $\mathbf{v} = (2, 0)$ as in Appendix A of part I, and the ordered, oriented Descartes configuration $\mathcal{D}_0^{'}$ is a permutation of $\mathcal{D}_0$. Then the geometric super-packing associated to $\mathcal{A}^S[\mathcal{D}_0^{'}]$ is therefore identical with that of $\mathcal{A}^S[\mathcal{D}_0]$, and that of $\mathcal{A}^S[\mathfrak{t}_x(\mathcal{D}_0^{'})]$ translates it by $(2, 0)$. Thus the geometric packing is invariant under this translation. The argument for translation by $(0, 2)$ is similar.

(2) This geometric Descartes configuration $\mathcal{D}_0$ is invariant under the reflections $(x, y) \mapsto (-x, y)$, and $(x, y) \mapsto (x, -y)$ viewed as Möbius transformations. The effect of these transformations on the ordered, oriented Descartes configuration is to permute its rows. It follows as in (1) that the associated geometric Apollonian super-packings are identical.

To see that these motions generate the full group of Euclidean motions leaving the super-packing invariant, we observe that the full group acts discontinuously on the plane. This is because the image of the $(0, 0, 1, 1)$ configuration is either left fixed, or else it moves a distance of at least two in some direction, so that its circles do not overlap. Thus it must be contained in a crystallographic group whose translation subgroup is given by (1) above. Now the only possibilities are to extend the group by a subgroup of the finite point group of motions leaving $(0, 0)$ fixed (of order 8) leaving the lattice $\mathbb{Z}[(0, 2), (2, 0)]$ of translations invariant. Here (2) gives an extension of order 4. No larger extension occurs by observing that otherwise the image of the $(0, 0, 1, 1)$ and $(-1, 2, 2, 3)$ configurations at the origin would cross themselves. ∎

One can now check that the 8 primitive geometric strongly integral super-packings given by Theorem 5.1 are obtained from the standard strongly integral super-packing by 8 cosets of the Euclidean motions $(x, y) \mapsto (x + 1, y)$, $(x, y) \mapsto (x, y + 1)$ and $(x, y) \mapsto (y, x)$ with respect to the symmetries in Theorem 6.1. They are specified by the location and orientation of the $(0, 0, 1, 1)$ configuration.

Our next result shows that every primitive integral Descartes configuration with no curvature zero occurs inside the geometric standard strongly integral super-packing in a specified location.

Theorem 6.2. *In the geometric standard strongly integral super-packing, for each (unordered) primitive integral Descartes quadruple (a, b, c, d) except for $(0, 0, 1, 1)$, there exists a Descartes configuration having these curvatures, such that the center of its largest circle lies in the closed unit square $\{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. The location of the center of this largest circle is unique.*

Proof. To establish existence, we first show that a Descartes configuration of the curvatures occurs somewhere inside the standard integral super-packing. This holds because the super-packing generated by such a configuration is an integral super-packing with divisor 1, which by Theorem 4.1 is unique up to a Euclidean motion. Thus the standard strongly integral super-packing must contain an isometric copy of it. Now that we have such a configuration inside the packing, we can use the translation symmetries in Theorem 6.1 to move it so that its largest circle has center inside the half-open square $\{(x, y) : -1 \le x < 1, -1 \le y < 1\}$. If we have $-1 \le x < 0$ then we apply the symmetry $(x, y) \mapsto (-x, y)$, while if $-1 \le y < 0$ we apply $(x, y) \mapsto (x, -y)$, as necessary.

To establish uniqueness, we argue by contradiction. If uniqueness failed, there would exist a Euclidean motion taking one of these Descartes configurations to the other. Since a single Descartes configuration generates the entire super-packing, we conclude that the super-packing is left invariant under this extra Euclidean motion. Theorem 6.1 described all such automorphisms and all of them, except the identity, map every point in the interior of the unit square square strictly outside the square. This contradicts the assumption that the center of the largest circle of the first configuration is mapped to that of the second, when at least one of

these points is strictly inside the unit square. In the remaining cases where both centers lie on the boundary, one shows they are must lie the same boundary edge and that the automorphism leaves this edge fixed, so they are identical. Note that this argument shows in passing that the largest circle is unique, once $(0,0,1,1)$ is excluded. ∎

We come now to the main result of this section, which asserts that the geometric standard super-packing contains a copy of every integral Apollonian circle packing in a canonical way. The circles in the geometric standard super-packing can be foliated into a union of geometric Apollonian packings.

Theorem 6.3. *(1) Each circle in the standard super-packing with center inside the half-open unit square* $\{(x,y) : 0 \le x < 1,\ 0 \le y < 1\}$ *is the exterior boundary circle of a unique primitive integral Apollonian circle packing contained in the geometric standard integral super-packing.*

(2) Every primitive integral Apollonian circle packing, except for the packing $(0,0,1,1)$*, occurs exactly once in this list.*

Proof. (1) Let the circle C be given. Recall from the proof of Theorem 3.1 that there is a unique minimal length admissible sequence $\mathbf{U}_m\mathbf{U}_{m-1}\cdots\mathbf{U}_1[\mathcal{D}_0]$ of generators of the super-Apollonian group that yield a Descartes configuration $\mathcal{D}$ containing the given circle C, say in its j-th position. Admissibility requires that $\mathbf{U}_m = \mathbf{S}_j$ or $\mathbf{S}_i^{\perp}$ for some $i \neq j$. Then multiplying by $\mathbf{S}_j^{\perp}$ also gives an admissible sequence, and the Descartes configuration

$$\mathcal{D}' := \mathbf{S}_j^{\perp}[\mathcal{D}] = \mathbf{S}_j^{\perp}\mathbf{U}_m\mathbf{U}_{m-1}\cdots\mathbf{U}_1[\mathcal{D}_0]$$

consists of the circle C plus three new circles nested inside the interior of C. This Descartes configuration $\mathcal{D}'$ generates an Apollonian packing having the circle C as outer boundary, contained in the standard strongly integral super-packing. It is unique, because if there were a second Apollonian packing inside the bounding circle it would contain circles crossing those in the first packing, contradicting Theorem 3.1.

(2) Recall from [5, Sect. 3 and 4] that inside each integer Apollonian packing is a positively oriented Descartes configuration whose absolute values of curvatures (a,b,c,d) are minimal, which is called a *root quadruple*. Theorem 4.1 of [5] showed that aside from the root quadruple $(0,0,1,1)$ every root quadruple is of the form $a < 0 < b \le c \le d$. Root quadruples are characterized by satisfying the extra condition

$$a + b + c \ge d. \tag{6.2}$$

All the circles in the resulting Apollonian packing are contained inside a *bounding circle* of curvature $N = |a|$, i.e., radius $\frac{1}{N}$. Theorem 6.2 shows that for each root quadruple, with all curvatures nonzero there is a matching Descartes configuration whose largest circle has center inside the unit square and this largest circle is unique. The Apollonian packing contained inside this circle by is the integral packing with the given root quadruple, and it is unique by the result of (1). (In some cases, like $(-1,2,2,3)$ the root configuration is not unique, but the root quadruple and the packing itself are always unique.) Thus every primitive integer Apollonian circle packing, except $(0,0,1,1)$, occurs exactly once bounding circles having center in the closed unit square. ∎

The initial part of the standard super-packing to depth 200 inside the unit square is pictured in Figure 8 below.

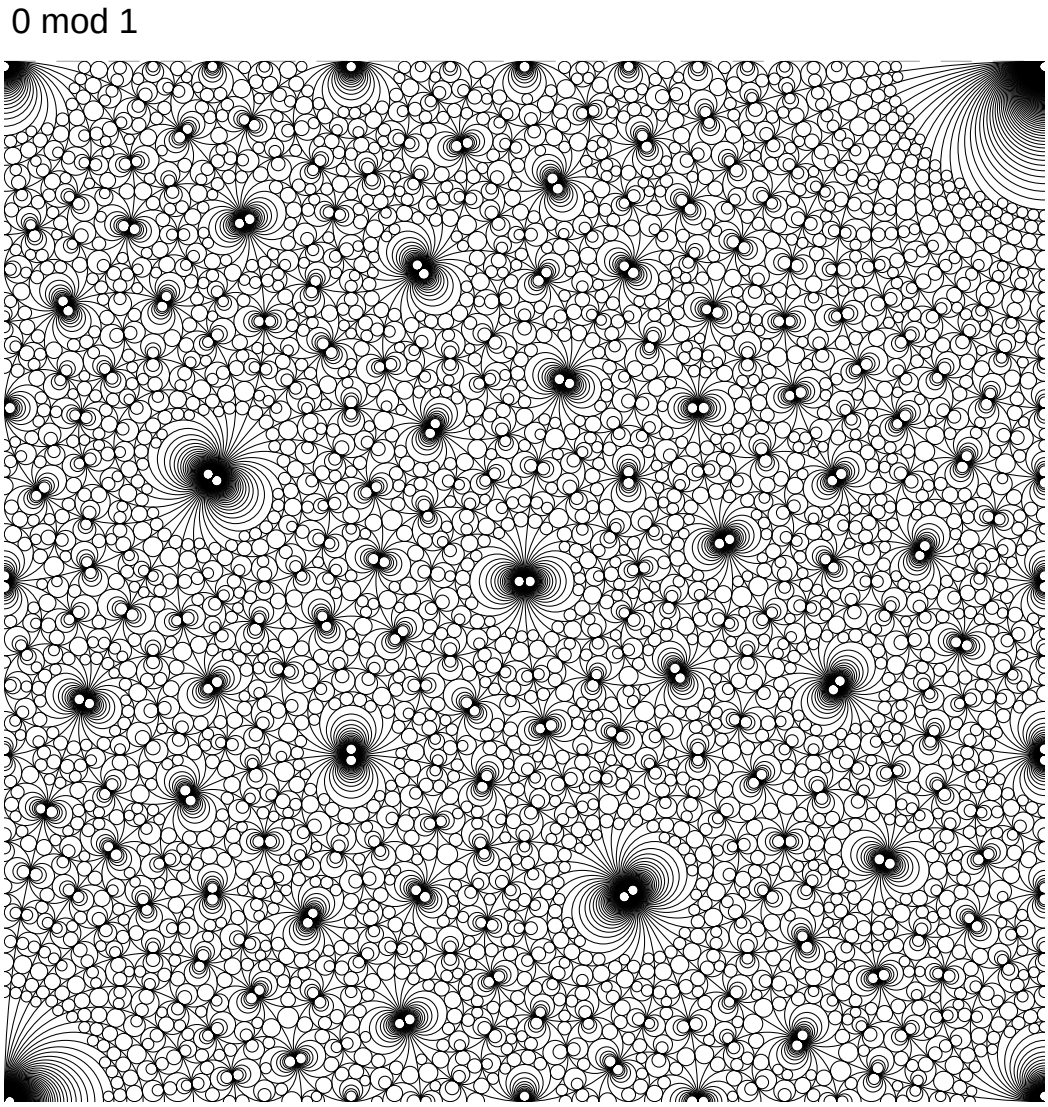

Figure 8: A "deeper" initial part of a super-packing (square of sidelength 1)

One can make further computer experiments plotting the circles having various curvatures restricted $(\bmod 4)$ inside the unit square. The results for circles having curvatures $1 \pmod 2$, $2 \pmod 4$ and $0 \pmod 4$ and size at most 200 are pictured in the following three figures.

The figures empirically indicate that the following extra reflection symmetries occur; the line of symmetry is indicated on the figure with a dotted line.

(a) $1 \pmod 2$ circles are symmetrical under reflection in the line $x = 1 - y$.

(b) $2 \pmod 4$ circles are symmetrical under reflection in the horizontal line $y = \frac{1}{2}$.

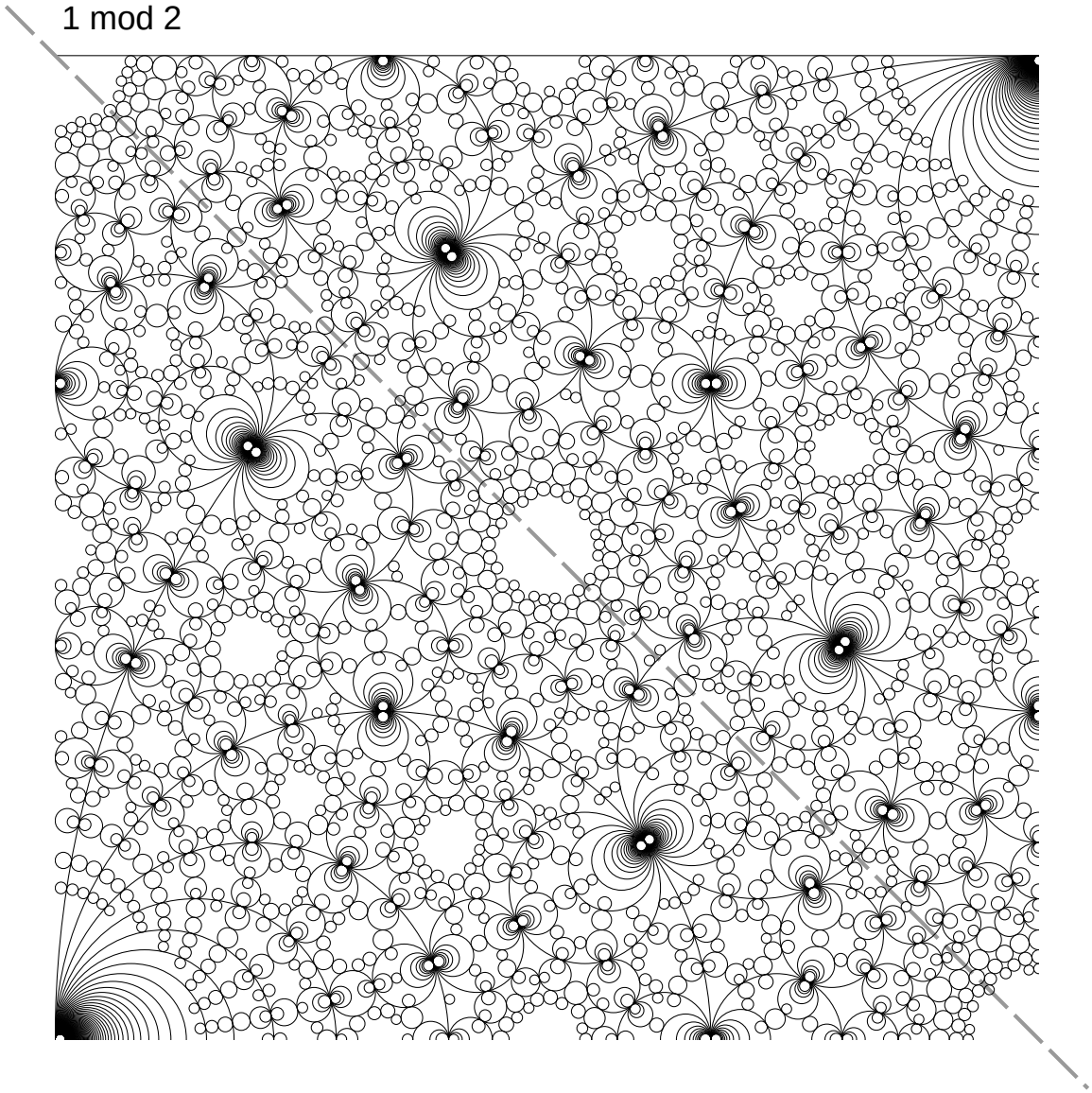

Figure 9: Circles of curvature 1 (mod 2)

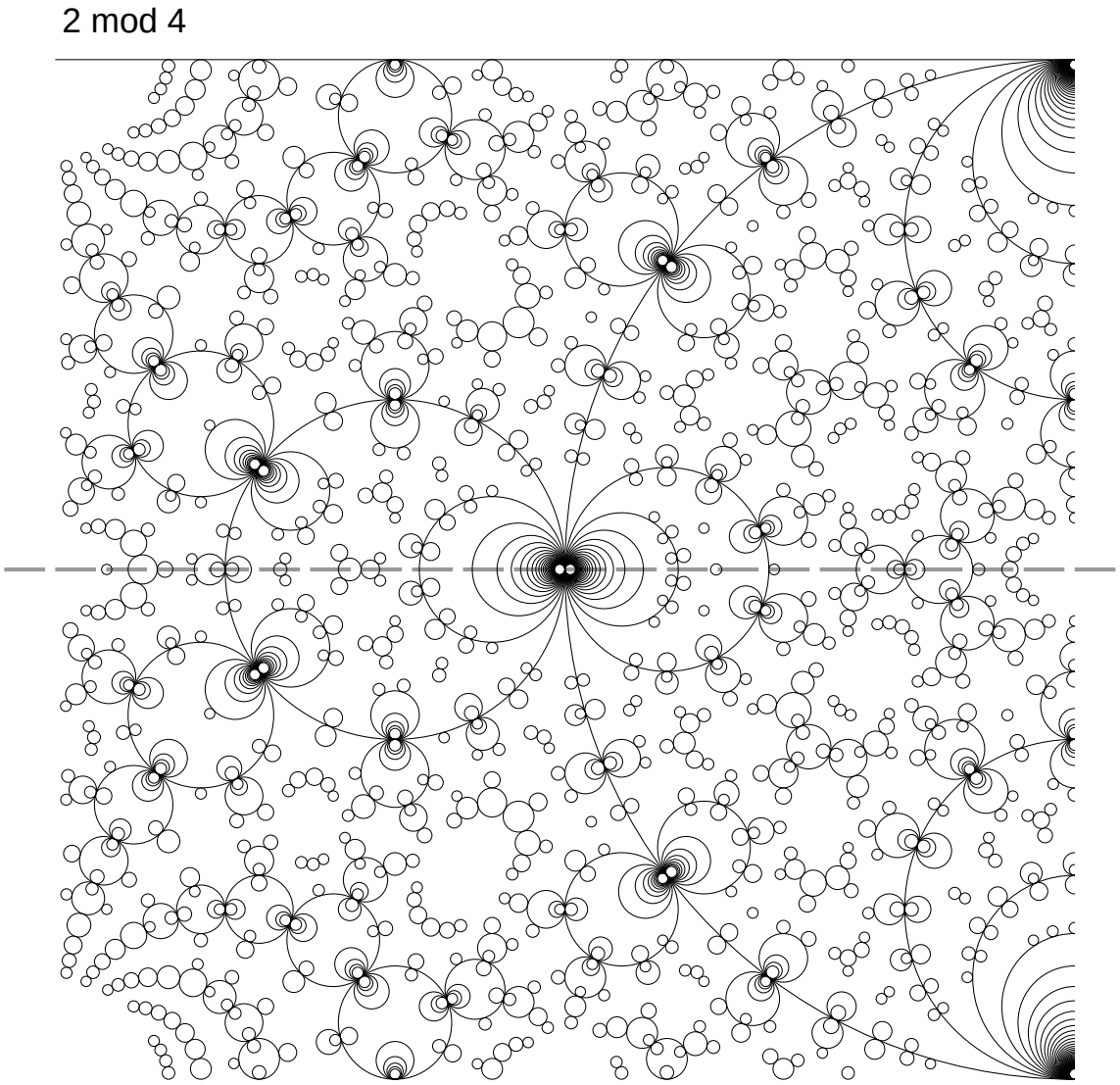

Figure 10: Circles of curvature 2 (mod 4)

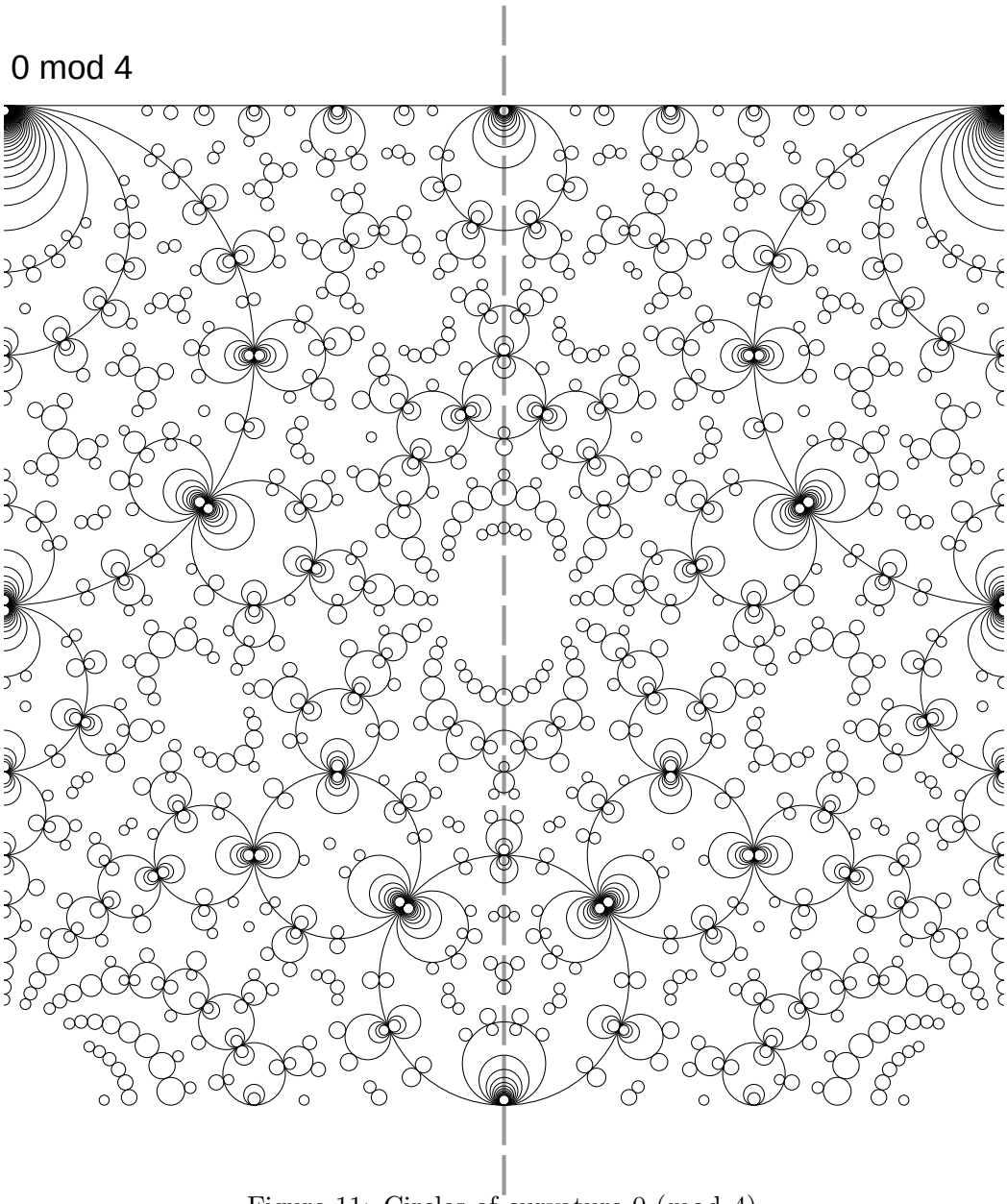

Figure 11: Circles of curvature 0 (mod 4)

(c) 0 (mod 4) circles are symmetrical under reflection in the vertical line $x = \frac{1}{2}$.

Based on this experimental evidence, one of the authors (CLM) conjectured that these symmetries hold. They were subsequently proved by S. Northshield [10].

We may also illustrate these symmetry properties at the level of circles of a fixed curvature. Recall from §3 that each such circle contains a unique Apollonian packing having it as outer circle. Such a circle can then be labelled by root quadruple in the sense of [5] of this integral Apollonian packing. The cases of curvature $-6, -8$, and -9 corresponding to the three cases above were pictured in §2. In our paper considering number-theoretic properties of integral Apollonian packings, it was shown in [5, Theorem 4.2] that the number of distinct primitive integral Apollonian packings with a given curvature $-n$ of the outer circle had an interpretation as a class number $h^{\pm}(-4n^2)$ of positive definite binary quadratic forms of discriminant $\Delta = -4n^2$, under $GL(2,\mathbb{Z})$-equivalence. One can raise the question whether there is some interpretation of the extra symmetries (a)-(c) above in terms of the associated class group structure under $SL(2,\mathbb{Z})$ equivalence.

7. The Super-Apollonian Group has Finite Covolume

In this section we show that the super-Apollonian group is of finite volume as a discrete subgroup of the real Lie group $Aut(Q_{\mathcal{D}}) \simeq O(3,1)$. This follows from the fact that the integer Lorentz group $O(3,1;\mathbb{Z})$ has finite covolume in $O(3,1)$, and the following result.

Theorem 7.1. *(1) The super-Apollonian group $\mathcal{A}^S$ is a normal subgroup of index 48 in the group $Aut(Q_D,\mathbb{Z})$. The group $Aut(Q_D,\mathbb{Z})$ is generated by the super-Apollonian group and the finite group of order 48 generated by the 4×4 permutation matrices and $\pm\mathbf{I}$.*

(2) The super-Apollonian group $\mathcal{A}^S$is a normal subgroup of index 96 in the group $G = \mathbf{J}_0\ O(3,1;\mathbb{Z})\mathbf{J}_0^{-1}$, where

$$\mathbf{J}_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The group G is generated by $Aut(Q_D,\mathbb{Z})$ and the duality matrix $\mathbf{D}$, and consists of matrices with integer and half-integer entries.

The duality matrix $\mathbf{D}$ is given by (5.3) in §5. This result allows us to associate to the super-Apollonian group the normal subgroup $\tilde{\mathcal{A}}^S := \mathbf{J}_0^{-1}\mathcal{A}^S\mathbf{J}_0$ of the integer Lorentz group $O(3,1;\mathbb{Z})$, of index 96.

The proof of Theorem 7.1 will be derived in a series of four lemmas. Let Γ be the group generated by adjoining to $\mathcal{A}$ the elements of the finite group of order 48 given by Perm_4 and $\pm I$, and let $\tilde{\Gamma} =< \Gamma, \mathbf{D} >$. The lemmas will prove that $\Gamma = Aut(Q_D,\mathbb{Z})$ and $\tilde{\Gamma} = \mathbf{J}_0\ O(3,1;\mathbb{Z})\mathbf{J}_0^{-1}$. Then analysis of cosets of $\mathcal{A}$ in these groups permits determining the index of $\mathcal{A}$ in these groups and showing normality.

To determine Γ and $\tilde{\Gamma}$ we will show

$$\Gamma \leq Aut(Q_D,\mathbb{Z}) \leq \tilde{\Gamma} = G.$$

It is easy to show that Γ is a subgroup of $\tilde{\Gamma}$ of index 2, and $\mathbf{D} \in \tilde{\Gamma}$ but $\mathbf{D} \notin Aut(Q_D, \mathbb{Z})$, so we get $Aut(Q_D, \mathbb{Z}) = \Gamma =< \mathcal{A}^S, \mathrm{Perm}_4, \pm\mathbf{I} >$. It is easy to check the inclusion $\Gamma \leq Aut(Q_D, \mathbb{Z})$ since the generators $\mathbf{S}_i, \mathbf{S}_i^T$, $\mathbf{P}_\sigma$ and $\pm\mathbf{I}$ of Γ are all in $Aut(Q_D, \mathbb{Z})$. The inclusion $Aut(Q_D, \mathbb{Z}) \leq G$ is proved in Proposition 7.2. The equation $G = \tilde{\Gamma}$ is proved in the next three lemmas. In Lemma 7.3 we prove that the integer Lorentz group is exactly the group $Aut(L_\mathbb{Z})$ of invertible linear transformations that leave the integral Lorentz cone $L_\mathbb{Z}$ invariant. Lemma 7.4 states that in order to show that a group $\mathcal{G}$ of invertible linear transformations of $L_\mathbb{Z}$ equals $Aut(L_\mathbb{Z})$, it is enough to show that (1) the action of G on $L_\mathbb{Z}$ is transitive, and (2) there exists a point $v \in L_\mathbb{Z}$ such that the stabilizer $\mathcal{S}_v = \{\mathbf{U} \in Aut(L_\mathbb{Z}) \mid \mathbf{U}v = v\}$ is a subset of $\mathcal{G}$. Using Lemma 7.4, we check that (1) the action of $\mathbf{J}_0\tilde{\Gamma}\mathbf{J}$ on $L_\mathbb{Z}$ is transitive, and (2), $\mathbf{J}_0\tilde{\Gamma}\mathbf{J}$ contains the stabilizer $\mathcal{S}_v$ of the point $v = (1, 1, 0, 0) \in L_\mathbb{Z}$. This proves $G = \tilde{\Gamma}$.

Lemma 7.2. *It is true that*

$$Aut(Q_D, \mathbb{Z}) \leq G = \mathbf{J}_0 O(3, 1; \mathbb{Z})\mathbf{J}_0^{-1}.$$

Proof. Let $\mathbf{U} \in Aut(Q_D, \mathbb{Z})$. We need to show that the entries of $\mathbf{J}_0^{-1}\mathbf{U}\mathbf{J}_0 = \mathbf{J}_0\mathbf{U}\mathbf{J}_0$ are all integers. Since $\mathbf{J}_0 = \frac{1}{2}\mathbf{1}\mathbf{1}^T - \mathbf{T}$, where

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

we have

$$\mathbf{J}_0\mathbf{U}\mathbf{J}_0 = \frac{1}{4}(\sum_{i,j} U_{ij})\mathbf{1}\mathbf{1}^T - \frac{1}{2}\mathbf{1}\mathbf{1}^T\mathbf{U}\mathbf{T} - \frac{1}{2}\mathbf{T}\mathbf{U}\mathbf{1}\mathbf{1}^T + \mathbf{T}\mathbf{U}\mathbf{T}, \tag{7.1}$$

where $\mathbf{T}\mathbf{U}\mathbf{T}$ is an integer matrix.

From $\mathbf{U}^T\mathbf{Q}_D\mathbf{U} = \mathbf{Q}_D$ and $\mathbf{Q}_D = \frac{1}{2}(2\mathbf{I} - \mathbf{1}\mathbf{1}^T)$, we have

$$\mathbf{U}^T(2\mathbf{I} - \mathbf{1}\mathbf{1}^T)\mathbf{U} = 2\mathbf{I} - \mathbf{1}\mathbf{1}^T. \tag{7.2}$$

Denote by $\mathbf{v}_i$ the i-th column of $\mathbf{U}$, and by $size(\mathbf{v}) := \mathbf{1}^T\mathbf{v}$ the size of a vector $\mathbf{v}$. Equating the entries of (7.2) we get $2\mathbf{v}_i \cdot \mathbf{v}_j - size(\mathbf{v}_i)size(\mathbf{v}_j) = 2\delta_{ij} - 1$. In particular, $size(\mathbf{v}_i)$, $i = 1, 2, 3, 4$ are odd integers. It follows that the matrix $\frac{1}{2}\mathbf{1}\mathbf{1}^T\mathbf{U}\mathbf{T}$ is integral.

Note that $Aut(Q_D, \mathbb{Z})$ is closed under transposition. This is because

$$\begin{aligned} &\mathbf{U} \in Aut(Q_D, \mathbb{Z}) \Longrightarrow \mathbf{U}^T\mathbf{Q}_D\mathbf{U} = \mathbf{Q}_D \Longrightarrow \mathbf{U}^T\mathbf{Q}_D\mathbf{U}\mathbf{Q}_D\mathbf{U}^T = \mathbf{U}^T \\ \Longrightarrow\ &\mathbf{U}\mathbf{Q}_D\mathbf{U}^T = (\mathbf{U}^T\mathbf{Q}_D)^{-1}\mathbf{U}^T = (\mathbf{Q}_D)^{-1} = \mathbf{Q}_D \Longrightarrow \mathbf{U}^T \in Aut(Q_D, \mathbb{Z}). \end{aligned} \tag{7.3}$$

Applying the same argument of the preceding paragraph to $\mathbf{U}^T$, we then prove that the matrix $\frac{1}{2}\mathbf{T}\mathbf{U}\mathbf{1}\mathbf{1}^T$ is integral.

Again using $2\mathbf{v}_i \cdot \mathbf{v}_j - size(\mathbf{v}_i)size(\mathbf{v}_j) = 2\delta_{ij} - 1$, summing over $i, j = 1, \ldots, 4$, we get

$$(\sum_{i,j} U_{i,j})^2 = 2(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) + 8.$$

Note that $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4=(size(\mathbf{r}_1),size(\mathbf{r}_2),size(\mathbf{r}_3),size(\mathbf{r}_4))^T$ where $\mathbf{r}_i$ is the i-th row of the matrix $\mathbf{U}$. The sum $\sum_i(size(\mathbf{r}_i))^2$ is a multiple of 4 since each $size(\mathbf{r}_i)$ is odd. It follows that $(\sum_{i,j}U_{i,j})^2$ is a multiple of 8. Since $\sum_{i,j}U_{i,j}$ is an integer, we conclude that $\sum_{i,j}U_{i,j}$ is a multiple of 4 and the matrix $\frac{1}{4}(\sum_{i,j}U_{ij})\mathbf{1}\mathbf{1}^T$ is integral. This proves Lemma 7.2. ∎

Clearly $\mathcal{A}^S$, Perm_4 and $\pm\mathbf{I}$ are subgroups of $Aut(Q_D,\mathbb{Z})$. Let Γ be the group generated by $\mathcal{A}^S$, Perm_4 and $\pm\mathbf{I}$, and let $\tilde{\Gamma}:=\langle\Gamma,\mathbf{D}\rangle$. Then Γ is a subgroup of $\tilde{\Gamma}$ of index 2.

The *Lorentz light cone* is the set of points $\{(y_0,y_1,y_2,y_3)^T\in\mathbb{R}^4\ :\ -y_0^2+y_1^2+y_2^2+y_3^2=0\}$. Let $L_{\mathbb{Z}}$ be the set of integer points in the Lorentz light cone, i,e,

$$L_{\mathbb{Z}}:=\{(y_0,y_1,y_2,y_3)^T\in\mathbb{Z}^4\ :\ \ -y_0^2+y_1^2+y_2^2+y_3^2=0\},$$

and let $Aut(L_{\mathbb{Z}})$ be the set of linear transformations that leave $L_{\mathbb{Z}}$ invariant.

Lemma 7.3. $Aut(L_{\mathbb{Z}})=O(3,1,\mathbb{Z})$.

Proof. It is clear that $O(3,1;\mathbb{Z})\subseteq Aut(L_{\mathbb{Z}})$. To show the other direction, let $\mathbf{U}\in Aut(L_{\mathbb{Z}})$. For any integer point $\mathbf{v}\in L_{\mathbb{Z}}$, $U\mathbf{v}\in L_{\mathbb{Z}}$. Therefore $(\mathbf{U}\mathbf{v})^TQ_{\mathcal{L}}(\mathbf{U}\mathbf{v})=0$, i.e., $\mathbf{v}^T(\mathbf{U}^TQ_{\mathcal{L}}\mathbf{U})\mathbf{v}=0$. It is easy to check that the only symmetric matrices $\mathbf{Q}$ satisfying $\mathbf{v}^T\mathbf{Q}\mathbf{v}=0$ for all $\mathbf{v}\in L_{\mathbb{Z}}$ are of the form $\mathbf{Q}=diag[-a,a,a,a]$. Hence $\mathbf{U}^T\mathbf{Q}_{\mathcal{L}}\mathbf{U}=c^2\mathbf{Q}_{\mathcal{L}}$, where $c=det(\mathbf{U})$.

Let

$$\mathbf{X}=\begin{bmatrix}1&1&1&1\\1&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

Note that every column vector of $\mathbf{X}$ is an integer point in $L_{\mathbb{Z}}$. Let $\mathbf{Y}=\mathbf{U}\mathbf{X}$, and $\mathbf{Z}=\mathbf{U}^{-1}\mathbf{X}$. Since $\mathbf{U}\in Aut(L_{\mathbb{Z}})$, every column of $\mathbf{Y}$ and $\mathbf{Z}$ is also an integer point in $L_{\mathbb{Z}}$. Therefore $\det(\mathbf{Y})=c\det(\mathbf{X})=-2c$, and $c\det(\mathbf{Z})=\det(\mathbf{X})=-2$. For each $(y_0,y_1,y_2,y_3)^T\in L_{\mathbb{Z}}$, we have $y_0^2=y_1^2+y_2^2+y_3^2$. It follows that either y_0, y_1, y_2, y_3 are all even, or y_0 and exactly one of y_1, y_2, y_3 are even. In both cases, $\det(\mathbf{Y})$ and $\det(\mathbf{Z})$ are even. This forces $c=\pm1$. Hence $\mathbf{U}^T\mathbf{Q}_{\mathcal{L}}\mathbf{U}=\mathbf{Q}_{\mathcal{L}}$, i.e., $\mathbf{U}\in O(3,1)$.

Since $\mathbf{U}$ maps points $(1,\pm1,0,0)^T$, $(1,0,\pm1,0)^T$, $(1,0,0,\pm1)^T$ to integer points, if (a,b,c,d) is a row of $\mathbf{U}$, then $a\pm b$, $a\pm c$, $a\pm d\in\mathbb{Z}$. Thus there exist integers a', b', c', d' of the same parity such that $a=\frac{a'}{2}$, $b=\frac{b'}{2}$, $c=\frac{c'}{2}$, $d=\frac{d'}{2}$. However, by Equation (7.3) $\mathbf{U}\mathbf{Q}_{\mathcal{L}}\mathbf{U}^T=\mathbf{Q}_{\mathcal{L}}$. This implies $-a^2+b^2+c^2+d^2=\pm1$. Therefore $b'^2+c'^2+d'^2\equiv a'^2 \pmod 4$. Hence a', b', c', d' must all be even, which means a, b, c, d are integers, and $\mathbf{U}\in O(3,1;\mathbb{Z})$. ∎

Lemma 7.4. *Let $\mathcal{G}$ be a group of linear transformations that preserves $L_{\mathbb{Z}}$. If the action is transitive, and there exists a point $v\in L_{\mathbb{Z}}$ such that the stabilizer $\mathcal{S}_v=\{\mathbf{U}\in Aut(L_{\mathbb{Z}})|\mathbf{U}v=v\}\subseteq\mathcal{G}$, then $\mathcal{G}=Aut(L_{\mathbb{Z}})$.*

Proof. Clearly $\mathcal{G}\subseteq Aut(L_{\mathbb{Z}})$. For any $\mathbf{P}\in Aut(L_{\mathbb{Z}})$, assume $\mathbf{P}(v)=v'$. Since $\mathcal{G}$ acts transitively on $L_{\mathbb{Z}}$, there exists $\mathbf{G}\in\mathcal{G}$ such that $\mathbf{G}(v')=v$. That is, $\mathbf{G}\mathbf{P}(v)=v$. So $\mathbf{G}\mathbf{P}\in\mathcal{S}_v\subseteq\mathcal{G}$, and then $\mathbf{P}\in\mathbf{G}^{-1}\mathcal{G}=\mathcal{G}$. This proves $Aut(L_{\mathbb{Z}})\subseteq\mathcal{G}$. ∎

Lemma 7.5. $\tilde{\Gamma}=G=\mathbf{J}_0O(3,1;\mathbb{Z})\mathbf{J}_0^{-1}$.

Proof. By Lemma 7.3, it is sufficient to prove that

$$\mathbf{J}_0^{-1}\tilde{\Gamma}\mathbf{J}_0 = \mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0 = Aut(L_{\mathbb{Z}}). \tag{7.4}$$

It is straightforward to check that $\mathbf{J}_0\mathbf{S}_i\mathbf{J}_0$, $\mathbf{J}_0\mathbf{S}_i^T\mathbf{J}_0$, $\mathbf{J}_0\mathbf{P}_\sigma\mathbf{J}_0$ and $\mathbf{J}_0\mathbf{D}\mathbf{J}_0$ are integer matrices. In particular,

$$\mathbf{J}_0\mathbf{S}_1\mathbf{J}_0 = \begin{bmatrix} 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix},$$

$\mathbf{J}_0\mathbf{P}_{34}\mathbf{J}_0 = \mathbf{P}_{34}$ and $\mathbf{J}_0\mathbf{D}\mathbf{J}_0 = diag[1,-1,-1,-1]$. Therefore $\mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0 \subseteq O(3,1;\mathbb{Z}) = Aut(L_{\mathbb{Z}})$.

For any integer point $(y_0,y_1,y_2,y_3) \in L_{\mathbb{Z}}$, $-y_0^2+y_1^2+y_2^2+y_3^2 = 0$ implies $y_0+y_1+y_2+y_3 \equiv 0 \pmod 2$. Hence $\mathbf{J}_0(y_0,y_1,y_2,y_3)^T$ is integral. It follows that

$$\mathbf{J}_0(L_{\mathbb{Z}}) = \{(a_1,a_2,a_3,a_4) \in \mathbb{Z}^4 \ : \ (a_1,a_2,a_3,a_4) \text{ are curvatures of a Descartes configuration }\}.$$

Then Theorem 4.2 implies that $\mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0$ acts transitively on the integral Lorentz light cone $L_{\mathbb{Z}}$, where $-\mathbf{I}$ exchanges the total orientation of a point in $L_{\mathbb{Z}}$.

We use Lemma 7.4 to prove equation (7.4) with $\mathcal{G} = J_0\tilde{\Gamma}J_0$. Let $\mathbf{v} := (1,1,0,0) \in L_{\mathbb{Z}}$ and consider its stabilizer

$$\mathcal{S}_{\mathbf{v}} := \{\mathbf{U} \in O(3,1;\mathbb{Z}) \ : \ \mathbf{U}\mathbf{v} = \mathbf{v}, \ \mathbf{U}^T\mathbf{Q}_{\mathcal{L}}\mathbf{U} = \mathbf{Q}_{\mathcal{L}}\}.$$

Assume $\mathbf{U} = (u_{i,j})_{i,j=1}^4$. Solving the equations

$$\mathbf{U}(1,1,0,0)^T = (1,1,0,0), \qquad \mathbf{U}^T\mathbf{Q}_{\mathcal{L}}\mathbf{U} = \mathbf{Q}_{\mathcal{L}}, \tag{7.5}$$

we obtain the following linear and quadratic relations between the entries of $\mathbf{U}$:

$$\begin{aligned} u_{12} &= 1-u_{11}, & u_{13} &= u_{23}, & u_{14} &= u_{24}, \\ u_{22} &= 1-u_{21}, & u_{32} &= -u_{31}, & u_{42} &= -u_{41}, \end{aligned}$$

and

$$\begin{aligned} u_{33}^2+u_{43}^2 = u_{34}^2+u_{44}^2 &= 1, & u_{33}u_{34}+u_{43}u_{44} &= 0, \\ u_{13} &= u_{31}u_{33}+u_{41}u_{43}, & u_{14} &= u_{31}u_{34}-u_{41}u_{44}, \\ u_{21} &= \frac{1}{2}(u_{31}^2+u_{41}^2). \end{aligned}$$

It follows that the matrix $\mathbf{U}$ can be expressed as

$$\mathbf{U} = \begin{pmatrix} 1+t & -t & gm+hn & km+ln \\ t & 1-t & gm+hn & km+ln \\ m & -m & g & k \\ n & -n & h & l \end{pmatrix}, \tag{7.6}$$

where $t = (m^2+n^2)/2$, $g^2+h^2 = k^2+l^2 = 1$, and $gk+hl = 0$. Since $g,h,k,l \in \mathbb{Z}$, we must have $(g,h),(k,l) \in \{(\pm1,0),(0,\pm1)\}$.

We can classify the matrices of the form (7.6) into four types, up to a possible multiplication by $\mathbf{P}_{34}$) , as follows.

$$\text{Type I}: \begin{pmatrix} 1+t & -t & m & n \\ t & 1-t & m & n \\ m & -m & 1 & 0 \\ n & -n & 0 & 1 \end{pmatrix}, \qquad \text{Type II}: \begin{pmatrix} 1+t & -t & m & -n \\ t & 1-t & m & -n \\ m & -m & 1 & 0 \\ n & -n & 0 & -1 \end{pmatrix},$$

$$\text{Type III}: \begin{pmatrix} 1+t & -t & -m & n \\ t & 1-t & -m & n \\ m & -m & -1 & 0 \\ n & -n & 0 & 1 \end{pmatrix}, \qquad \text{Type IV}: \begin{pmatrix} 1+t & -t & -m & -n \\ t & 1-t & -m & -n \\ m & -m & -1 & 0 \\ n & -n & 0 & -1 \end{pmatrix},$$

where $t = (m^2+n^2)/2$ and $m, n, t \in \mathbb{Z}$.

We denote a matrix of type X with parameters m, n by $\mathbf{U}(m, n; \mathrm{X})$. The following equations can be easily checked.

$$\begin{aligned} \mathbf{U}(m,n;\mathrm{I})\mathbf{U}(k,l;\mathrm{I}) &= \mathbf{U}(m+k,n+l;\mathrm{I}), \\ \mathbf{U}(m,n;\mathrm{II})\mathbf{U}(k,l;\mathrm{I}) &= \mathbf{U}(m+k,n-l;\mathrm{II}), \\ \mathbf{U}(m,n;\mathrm{III})\mathbf{U}(k,l;\mathrm{I}) &= \mathbf{U}(m-k,n+l;\mathrm{III}), \\ \mathbf{U}(m,n;\mathrm{IV})\mathbf{U}(k,l;\mathrm{I}) &= \mathbf{U}(m-k,n-l;\mathrm{IV}). \end{aligned}$$

Also we have

$$\mathbf{U}(1,1;\mathrm{I})^{-1} = \mathbf{U}(-1,-1;\mathrm{I}), \qquad \mathbf{U}(1,-1;\mathrm{I})^{-1} = \mathbf{U}(-1,1;\mathrm{I}).$$

Therefore the stabilizer $\mathcal{S}_v$ is generated by $\mathbf{P}_{34}$, $\mathbf{U}(0,0;\mathrm{I})$, $\mathbf{U}(0,0;\mathrm{II})$ $\mathbf{U}(0,0;\mathrm{III})$, $\mathbf{U}(0,0;\mathrm{IV})$ together with $\mathbf{A} = \mathbf{U}(1,1;\mathrm{I}), \mathbf{B} = \mathbf{U}(1,-1;\mathrm{I})$.

Note that $\mathbf{U}(0,0;\mathrm{I}) = \mathbf{U}(0,0;\mathrm{II})^2$, $\mathbf{U}(0,0;\mathrm{III}) = \mathbf{P}_{34}\mathbf{U}(0,0;\mathrm{II})\mathbf{P}_{34}$, and $\mathbf{U}(0,0;\mathrm{IV}) = \mathbf{U}(0,0;\mathrm{II})\mathbf{U}(0,0;\mathrm{III})$. Thus $\mathcal{S}_v$ is generated by the matrices $\mathbf{P}_{34}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C} = \mathbf{U}(0,0;\mathrm{II})$.

Now one can check that

$$\begin{aligned} \mathbf{P}_{34} &= \mathbf{J}_0\mathbf{P}_{34}\mathbf{J}_0 \in \mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0, \\ \mathbf{C} &= \mathrm{diag}[1,1,1,-1] = \mathbf{J}_0(-\mathbf{I}\mathbf{P}_{23}\mathbf{P}_{14}\mathbf{D})\mathbf{J}_0 \in \mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0, \\ \mathbf{A} &= (\mathbf{J}_0\mathbf{S}_1\mathbf{J}_0)\mathrm{diag}[1,1,-1,-1] = \mathbf{J}_0(\mathbf{S}_1\mathbf{P}_{12}\mathbf{P}_{34})\mathbf{J}_0 \in \mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0, \\ \mathbf{B} &= \big(\mathbf{J}_0\mathbf{P}_{34}\mathbf{J}_0\big)\mathbf{C}\big(\mathbf{J}_0\mathbf{S}_2\mathbf{J}_0\big)\mathbf{C} \in \mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0. \end{aligned}$$

Hence $\mathcal{S}_v \subseteq \mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0$. By Lemma 7.4 $\mathbf{J}_0\tilde{\Gamma}\mathbf{J}_0 = Aut(L_{\mathbb{Z}}) = O(3,1;\mathbb{Z})$, or equivalently, $\tilde{\Gamma} = \mathbf{J}_0 O(3,1;\mathbb{Z})\mathbf{J}_0^{-1}$. This finishes the proof. ∎

Proof of Theorem 7.1. Lemma 7.2 and Lemma 7.5 show that $\Gamma \leq Aut(Q_D, \mathbb{Z}) \leq G$. Hence $\Gamma = Aut(Q_D, \mathbb{Z})$ since Γ is a subgroup of G of index 2 and the duality matrix $\mathbf{D}$ is not in $Aut(Q_D, \mathbb{Z})$. In other words, $Aut(Q_D, \mathbb{Z})$ is generated by the super-Apollonian group $\mathcal{A}^S$ and the finite group of order 48 generated by the 4×4 permutation matrices and $\pm I$. The

super-Apollonian group $\mathcal{A}^S$ is a normal subgroup of $Aut(Q_D, \mathbb{Z})$, since $\mathbf{P}_\sigma \mathbf{S}_i \mathbf{P}_{\sigma^{-1}} = \mathbf{S}_{\sigma(i)}$, and $\mathbf{P}_\sigma \mathbf{S}_i^T \mathbf{P}_{\sigma^{-1}} = \mathbf{S}_{\sigma(i)}^T$. The index is 48 since $\mathcal{A}^S \cap (\mathrm{Perm}_4 \times \{\pm \mathbf{I}\}) = \mathbf{I}$, (c.f. §5 of Part I [3]).

By Lemma 7.5, the group G is generated by $Aut(Q_D, \mathbb{Z})$ and $\mathbf{D}$. Note that $\mathbf{D}^2 = I$ and $\mathbf{D}\mathbf{S}_i\mathbf{D} = \mathbf{S}_i^T$. It follows that the super-Apollonian group is a normal subgroup of G with index 96. ■

The second part of Theorem 7.1 can be rephrased as asserting that the conjugate group $\tilde{\mathcal{A}}^S = \mathbf{J}_0 \mathcal{A} \mathbf{J}_0$ is a normal subgroup of index 96 in $O(3,1)$. Its generators $\tilde{\mathbf{S}}_j = \mathbf{J}_0 \mathbf{S}_j \mathbf{J}_0^{-1}$ and $\tilde{\mathbf{S}}_j^\perp = \mathbf{J}_0 \mathbf{S}_j^\perp \mathbf{J}_0^{-1}$ are given by

$$\tilde{\mathbf{S}}_1 \begin{bmatrix} 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{S}}_2 = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

and

$$\tilde{\mathbf{S}}_3 = \begin{bmatrix} 2 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{S}}_4 = \begin{bmatrix} 2 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The generators $\tilde{\mathbf{S}}_j^\perp = \mathbf{J}_0 \mathbf{S}_j^\perp \mathbf{J}_0^{-1}$ are given by

$$\tilde{\mathbf{S}}_j^\perp = (\tilde{\mathbf{S}}_j)^T,$$

which follows using $\mathbf{S}_j^\perp = \mathbf{S}_j^T$ and $\mathbf{J}_0 = \mathbf{J}_0^T = \mathbf{J}_0^{-1}$.

8. Super-Integral Super-Packings

This section treats the strongest form of integrality for super-packings, which is that where one (and hence all) Descartes configurations in the super-packing have an integral augmented curvature-center coordinate matrix $\mathbf{W}_\mathcal{D}$. We say that a Descartes configuration with this property is *super-integral*, and the same for the induced super-Apollonian packing. The following result classifies such packings.

Theorem 8.1. *(1) These are exactly* 14 *different geometric super-packings that are super-integral.*

(2) The set of ordered, oriented Descartes configurations that are super-integral comprise 672 *orbits of the super-Apollonian group.*

These packings are classified here as rigid objects, not movable by Euclidean motions. To prove this result, it suffices to determine which strongly-integral configurations are super-integral. The next result classifies the possible types of super-integral Descartes configurations, according to the allowed value of their divisors.

Theorem 8.2. *Suppose that an ordered, oriented Descartes configuration $\mathcal{D}$ in $\mathbb{R}^2$ has integral curvature-center coordinates $\mathbf{M} = \mathbf{M}_{\mathcal{D}}$, and let $g = \gcd(a_1, a_2, a_3, a_4)$, where (a_1, a_2, a_3, a_4) is its first column of signed curvatures. Then $\mathcal{D}$ has integral augmented curvature-center coordinates $\mathbf{W}_{\mathcal{D}}$ if and only if one of the following conditions hold.*

(i) $g = 1$, or

(ii) $g = 2$, and each row of $\mathbf{M}$ has the sum of its last two entries being $1 \pmod 2$.

(iii) $g = 4$, and the last columns rows of $\mathbf{M}$ are congruent to

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \pmod 2.$$

Proof. By Theorem 5.2, there exists a matrix $\mathbf{U} \in \mathcal{A}^S$ and a permutation matrix $\mathbf{P}$ such that

$$\mathbf{PUM}_{\mathcal{D}} = A_{m,n}[g] \text{ or } B_{m,n}[g],$$

for some $m, n \in \{0, 1\}$, where $A_{m,n}[g]$ and $B_{m,n}[g]$ are given in Formula (5.1), and their corresponding augmented curvature-center coordinate matrices are $\tilde{A}_{m,n}[g]$, $\tilde{B}_{m,n}[g]$, given in Formula (5.2). Since each generator of $\mathcal{A}^S$ preserves the super-integrality, as well as the parity of every element of $\mathbf{W}_{\mathcal{D}}$, it follows that $\mathbf{W}_{\mathcal{D}}$ is integral if and only if one of the following conditions holds:

1. $g = 1$, or
2. $g = 2$, and $m^2 + n^2 - 1 \equiv 0 \pmod 2$, or
3. $g = 4$, and the reduced form is $A_{0,1}[4]$ or $B_{1,0}[4]$, up to a permutation of rows.

In Case 2 we need $m + n \equiv 1 (\text{mod} 2)$; in Case 3 the condition is equivalent to the one stated in the theorem. ∎

Proof of Theorem 8.1. (1) Since each geometric super-packing corresponds to 48 distinct orbits of the super-Apollonian group on ordered, oriented Descartes configurations, to show there are exactly 14 geometric super-packings, it suffices to show there are exactly 672 orbits of the super-Apollonian group that are super-integral, which is (2).

(2)Theorems 5.2 and 8.2 allow us to classify the set of ordered, oriented Descartes configurations that are super-integral by the action of $\mathcal{A}^S$. From the criterion of Theorem 8.2, we have:

1. For $g = 1$, any strongly integral Descartes configuration is super-integral.
2. For $g = 2$, half of those orbits are super-integral, namely, those whose reduced forms are $A_{0,1}[2], A_{1,0}[2], B_{0,1}[2]$ or $B_{1,0}[2]$, up to a permutation of rows.
3. For $g = 4$, one fourth of those orbits are super-integral, namely, those whose reduced form are $A_{0,1}[4]$ or $B_{1,0}[4]$, up to a permutation of rows.
4. For $g \neq 1, 2, 4$, there are no super-integral Descartes configurations.

$\mathbf{g}$	# of orbits of $\mathcal{A}^S$	Representative
$(1,1,1,1)$	96	$A_{1,1}[1]$, $B_{1,1}[1]$
$(2,1,1,1)$	96	$A_{1,0}[1]$, $B_{0,1}[1]$
$(1,1,2,1)$	48	$A_{0,0}[1]$
$(1,1,1,2)$	48	$B_{0,0}[1]$
$(4,1,2,1)$	48	$A_{0,1}[1]$
$(4,1,1,2)$	48	$B_{1,0}[1]$
$(1,2,1,1)$	96	$A_{1,0}[2]$, $B_{0,1}[2]$
$(2,2,2,1)$	48	$A_{0,1}[2]$
$(2,2,1,2)$	48	$B_{1,0}[2]$
$(1,4,2,1)$	48	$A_{0,1}[4]$
$(1,4,1,2)$	48	$B_{1,0}[4]$

Table 1: Orbits of super-integral Descartes configurations classified by $\mathbf{g}$.

More details of this calculation are given in Table 1. To explain the notation in Table 1, for any 4×4 integral matrix $\mathbf{W}$, let g_i be the greatest common divisor of entries $w_{1,i}$, $w_{2,i}$, $w_{3,i}$, $w_{4,i}$ in the i-th column. Then the action of $\mathcal{A}^S$ preserves the vector $\mathbf{g} = (g_1, g_2, g_3, g_4)$. (For $\mathbf{W} = \mathbf{W}_{\mathcal{D}}$, g_2 is the greatest common divisor of the curvatures.) In each row of the table we give the number of orbits of $\mathcal{A}^S$ formed by the set of ordered, oriented Descartes configurations that are super-integral with the given $\mathbf{g}$. We also list the representatives of those orbits. Note that each entry in the column labelled "Representative" stands for 48 orbits, obtained by taking two choices of (total) orientation, and 24 choices of permutation of rows. Also note the symmetry that the configurations with $\mathbf{g}$ are inverse of the ones with $\mathbf{g}' = \mathbf{P}_{12}\mathbf{g}$. We conclude from the table that the set of ordered, oriented Descartes configurations that are super-integral comprise $384(1 + \frac{1}{2} + \frac{1}{4}) = 672$ orbits of the super-Apollonian group. ∎

9. Concluding remarks

This paper showed that the ensemble of all primitive, strongly integral Apollonian circle packings can be simultaneously described in terms of an orbit of a larger discrete group, the super-Apollonian group, acting on the standard strongly-integral super-packing. Study of the locations of the individual integer packings inside the standard super-packing, presented in §6, leads to interesting questions, not all of which are resolved. The standard super-packing also played a role in analyzing the structure of the super-Apollonian group as a discrete subgroup, carried out in §7.

The various illustrations show the usefulness of graphical representations, as a guide to both finding and illustrating results. This contribution is due in large part to the statistician co-authors (CLM and AW). Graphics were particularly useful in finding extra symmetries of these objects, such as those illustrated in Figures 9–11 and subsequently proved by S. Northshield [10]. However one must not forget the adage of H. M. Stark [12, p. 225]: "A picture is worth a thousand words, provided one uses another thousand words to justify the picture." Section 3 of this paper provides such a justification for certain features of Figure 4.

There remain some open questions, particularly concerning the classification of all integer root quadruples classified by fixed curvature $-N$ of the bounding circle. This quantity is known to be interpretable as a class number, as described in [5, Theorem 4.2]. In §6 we observed some symmetries of these root quadruples inside the standard super-packing, see Figures 4–6. There is a new invariant that can be associated to such quadruples, which is their nesting depth as defined in §4 with respect to the generating quadruple $\mathcal{D}_0$ of the standard super-packing. It would be interesting to see whether this invariant might give some further insight into class numbers.

10. Appendix. Strong Integrality Criterion

This appendix establishes that to show a Descartes configuration is strongly integral it suffices to show that three of its four circles are strongly integral. This affirmatively answers a question posed to us by K. Stephenson.

Theorem 10.1. *A Descartes configuration $\mathcal{D}$ has integral curvature-center coordinates $\mathbf{M}_{\mathcal{D}}$ if and only if it contains three circles having integer curvatures and whose curvature$\times$centers, viewed as complex numbers, lie in $\mathbb{Z}[i]$.*

Proof. The condition is clearly necessary, and the problem is to show it is sufficient. We write the circle centers as complex numbers $\mathbf{z}_j = x_j + iy_j$. So suppose $\mathcal{D}$ contains three circles with curvatures $b_1,\ b_2,\ b_3 \in \mathbb{Z}$ and with curvature$\times$centers $b_1\mathbf{z}_1,\ b_2\mathbf{z}_2,\ b_3\mathbf{z}_3 \in \mathbb{Z}[i]$. We must show that the fourth circle in the configuration has $b_4 \in \mathbb{Z}$ and $b_4\mathbf{z}_4 \in \mathbb{Z}[i]$.

For later use, we note that Theorem 3.1 of Part I [3] has a nice interpretation using complex numbers $\mathbf{z}$ to represent circle centers. This was formulated in [6] as the Complex Descartes theorem. It gives

$$b_1^2\mathbf{z}_1^2 + b_2^2\mathbf{z}_2^2 + b_3^2\mathbf{z}_3^2 + b_4^2\mathbf{z}_4^2 = \frac{1}{2}(b_1\mathbf{z}_1 + b_2\mathbf{z}_2 + b_3\mathbf{z}_3 + b_4\mathbf{z}_4)^2. \tag{10.1}$$

and

$$b_1^2\mathbf{z}_1 + b_2^2\mathbf{z}_2 + b_3^2\mathbf{z}_3 + b_4^2\mathbf{z}_4 = \frac{1}{2}(b_1\mathbf{z}_1 + b_2\mathbf{z}_2 + b_3\mathbf{z}_3 + b_4\mathbf{z}_4)(b_1 + b_2 + b_3 + b_4). \tag{10.2}$$

We claim that $b_4 \in \mathbb{Z}$. This is proved in the following two cases.

Case 1. $b_1b_2b_3 \neq 0$.

We first suppose that $\mathbf{z}_1 = 0$. If both x_2 and x_3 are zero, then $-\frac{1}{b_1} = \frac{1}{b_2} + \frac{1}{b_3}$, which means $b_1b_2 + b_2b_3 + b_3b_1 = 0$. Hence $b_4 = b_1 + b_2 + b_3 \in \mathbb{Z}$. Otherwise by permuting $\mathbf{z}_2$ and $\mathbf{z}_3$ if necessary we may assume that $x_2 \neq 0$. Then the following equations encode the distance between the circle centers, since the circles touch.

$$x_2^2 + y_2^2 = (\frac{1}{b_1} + \frac{1}{b_2})^2, \tag{10.3}$$

$$x_3^2 + y_3^2 = (\frac{1}{b_1} + \frac{1}{b_3})^2, \tag{10.4}$$

$$(x_3 - x_2)^2 + (y_3 - y_2)^2 = (\frac{1}{b_2} + \frac{1}{b_3})^2. \tag{10.5}$$

We wish to solve these equations for y_3 in terms of b_1, b_2, b_3, x_1 and x_2. To this end we subtract the first two equations from the third and obtain

$$2(x_2x_3 + y_2y_3) = (\frac{1}{b_1} + \frac{1}{b_2})^2 + (\frac{1}{b_1} + \frac{1}{b_3})^2 - (\frac{1}{b_2} + \frac{1}{b_3})^2 := R.$$

Calling the right side of this equation R, we obtain

$$x_3 = \frac{1}{2x_2}\left(R - 2y_2y_3\right).$$

where the division is allowed since $x_2 \neq 0$. Substituting this in the second equation yields a quadratic equation in y_3, with x_3 eliminated, namely (after multiplying by $4x_2^2$),

$$4(x_2^2 + y_2^2)y_3^2 - 4R\ y_2 y_3 + R^2 - 4x_2^2(\frac{1}{b_1} + \frac{1}{b_3})^2 = 0.$$

Since y_3 is rational, this equation has rational solutions, so the discriminant Δ must be the square of a rational number. After some calculation one obtains

$$\Delta = 16^2 x_2^2 \left(\frac{b_1 b_2 + b_1 b_3 + b_2 b_3}{(b_1 b_2 b_3)^2} \right).$$

Since x_2, b_1, b_2 are nonzero it follows that $b_1 b_2 + b_2 b_3 + b_1 b_3$ is a perfect square. Viewing the Descartes equation as a quadratic equation in b_4 we obtain the formula $b_4 := b_1 + b_2 + b_3 \pm 2\sqrt{b_1 b_2 + b_2 b_3 + b_1 b_3}$, which shows that both roots are integers. These are the oriented curvatures of the two possible choices for the fourth circle in the Descartes configuration, so $b_4 \in \mathbb{Z}$.

Now assume that $\mathbf{z}_1 = x_1 + iy_1$ is arbitrary. Define

$$(s_2, t_2) := (x_2 - x_1, y_2 - y_1) \quad \text{and} \quad (s_3, t_3) := (x_3 - x_1, y_3 - y_1).$$

Then s_2, t_2, s_3, t_3 are rational numbers, and they also satisfy equations (10.3), (10.4), (10.5). (just replace x_2, y_2, x_3, y_3 in (10.3), (10.4), (10.5) by s_2, t_2, s_3, t_3, respectively.) By the preceding argument, we again have $16s_2^2(b_1 b_2 + b_2 b_3 + b_1 b_3)/(b_1 b_2 b_3)^2 = q^2$ for some rational number q. This implies that $b_1 b_2 + b_2 b_3 + b_1 b_3$ is a perfect square and consequently that b_4 is an integer.

Case 2. $b_1 b_2 b_3 = 0$. This is proved similarly, with an easier calculation.

The claim follows, so $b_4 \in \mathbb{Z}$.

We now proceed to show that $b_4 \mathbf{z}_4 \in \mathbb{Z}[i]$. Now (10.1) gives

$$b_4 \mathbf{z}_4 = b_1 \mathbf{z}_1 + b_2 \mathbf{z}_2 + b_3 \mathbf{z}_3 \pm 2\sqrt{b_1 \mathbf{z}_1 b_2 \mathbf{z}_2 + b_2 \mathbf{z}_2 b_3 \mathbf{z}_3 + b_1 \mathbf{z}_1 b_3 \mathbf{z}_3}. \tag{10.6}$$

The equation (10.2) gives

$$(b_1 + b_2 + b_3 - b_4) b_4 \mathbf{z}_4 = 2(b_1^2 \mathbf{z}_1 + b_2^2 \mathbf{z}_2 + b_3^2 \mathbf{z}_3) - (b_1 + b_2 + b_3 + b_4)(b_1 \mathbf{z}_1 + b_2 \mathbf{z}_2 + b_3 \mathbf{z}_3). \tag{10.7}$$

We treat two mutually exhaustive cases.

Case 1. $b_1 + b_2 + b_3 \neq b_4$.

Then (10.7) gives $b_4 \mathbf{z}_4 = x_4 + iy_4$ for some rational numbers x_4, y_4. However $b_1 \mathbf{z}_1, b_2 \mathbf{z}_2, b_3 \mathbf{z}_3$ are integers by hypothesis, whence (10.6) shows that $b_4 z_4$ is an algebraic integer. Since x_4, y_4 are rational, we conclude that $b_4 \mathbf{z}_4$ must be an integer, i.e., in $\mathbb{Z}[i]$.

Case 2. $b_1 + b_2 + b_3 = b_4$. In this case, we have $b_1 b_2 + b_2 b_3 + b_1 b_3 = 0$. Now from (10.7), we have

$$b_1^2 \mathbf{z}_1 + b_2^2 \mathbf{z}_2 + b_3^2 \mathbf{z}_3 = (b_1 + b_2 + b_3)(b_1 \mathbf{z}_1 + b_2 \mathbf{z}_2 + b_3 \mathbf{z}_3),$$

which can be simplified to

$$b_2 b_3 \mathbf{z}_1 + b_1 b_3 \mathbf{z}_2 + b_1 b_2 \mathbf{z}_3 = 0.$$

Thus

$$\begin{aligned}
(b_1\mathbf{z}_1)(b_2\mathbf{z}_2) + (b_2\mathbf{z}_2)(b_3\mathbf{z}_3) + (b_3\mathbf{z}_3)(b_1\mathbf{z}_1) &= b_1(b_2\mathbf{z}_2 + b_3\mathbf{z}_3)\frac{b_1b_3\mathbf{z}_2 + b_1b_2\mathbf{z}_3}{-b_2b_3} + (b_2\mathbf{z}_2)(b_3\mathbf{z}_3) \\
&= -b_1^2\mathbf{z}_2^2 - b_1^2\mathbf{z}_3^2 + \mathbf{z}_2\mathbf{z}_3\left(b_2b_3 - \frac{b_1^2}{b_2b_3}(b_2^2 + b_3^2)\right) \\
&= -b_1^2(\mathbf{z}_2 - \mathbf{z}_3)^2.
\end{aligned}$$

Now $b_1(\mathbf{z}_2 - \mathbf{z}_3)$ is a Gaussian rational number whose square is integral. Hence $b_1(\mathbf{z}_2 - \mathbf{z}_3)$ must be integral. It follows that $b_4\mathbf{z}_4 = b_1\mathbf{z}_1 + b_2\mathbf{z}_2 + b_3\mathbf{z}_3 \pm 2b_1(\mathbf{z}_2 - \mathbf{z}_3)i$ is in $\mathbb{Z}[i]$. ∎

Since the Apollonian group consists of integer matrices, all Descartes configurations in an Apollonian packing generated by a strongly integral Descartes configuration are strongly integral. This explains the integrality properties of curvatures and curvature × center pictured in the packing in §1, for example. The previous result now gives a weaker necessary and sufficient condition for an Apollonian packing to be strongly integral.

Theorem 10.2. *An Apollonian circle packing is strongly integral if and only if it contains three mutually tangent circles which have integer curvature-center coordinates.*

Proof. Suppose we are give three mutually tangent circles in the packing that are strongly integral. Any set of three mutually tangent circles in the packing is part of some Descartes configuration in this packing. This follows from the recursive construction of the packing, which has a finite number of circles at each iteration. If iteration j is the first iteration at which all three tangent circles are present, at that iteration they necessarily belong to a unique Descartes configuration. Theorem 10.1 now implies that this Descartes configuration is strongly integral. It then follows that the whole Apollonian packing is strongly integral. This proves the "if" direction, and the "only if" direction is immediate. ∎

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```
email:  graham@ucsd.edu
        lagarias@umich.edu
        colinm@research.avayalabs.com
        allan@research.att.com
        cyan@math.tamu.edu
```