ET 438a
Automatic Control Systems Technology
Laboratory 4
Practical Differentiator Response

Objective: Design a practical differentiator circuit using common OP AMP circuits. Test the frequency response and phase shift of the differentiator with a variable frequency sine wave signal. Compare the lab measurements to the theoretical calculations for the circuit to check the design. Observe the differentiator output signals for various types of input signals commonly used in lab.

Theoretical Background

The mathematical operation of differentiation can be simulated by removing the input resistor in an inverting OP AMP circuit and inserting a capacitor. This ideal differentiator circuit is show in Figure 1.

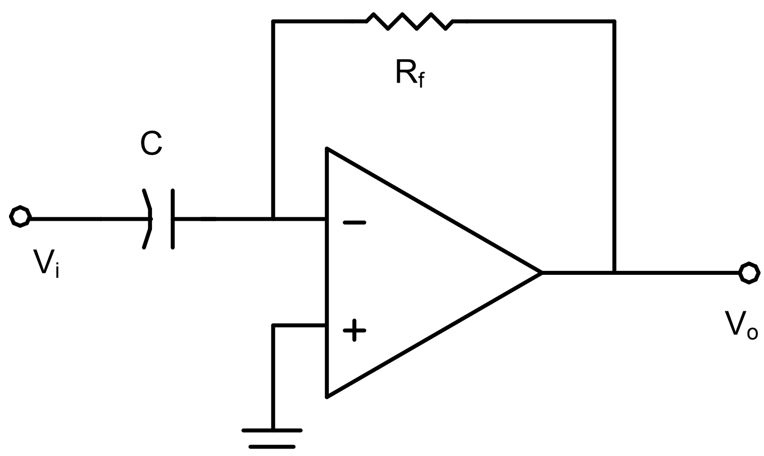

Figure 1. Ideal Differentiator Circuit.

If ideal OP AMP circuit operation is assumed, no current will flow into the inverting terminal of the amplifier due to the infinite input impedance. Also, the voltage between the inverting and non-inverting terminal is equal due to the effects of the negative feedback. This means that the voltage at the inverting terminal is at ground potential. So:

$$-I_f = I_C$$

The current in the capacitor is given by:

$$i_c = C\frac{dv}{dt} \qquad (1)$$
$$V_o = -\ i_f \bullet R_f$$

Combining the last three equations above gives the input-output relationship of the ideal differentiator circuit.

$$V_o(t) = -RC\ \frac{d\,V_i(t)}{dt} \quad (a) \qquad (2)$$
$$-\ RC = K_d \quad (b)$$

The constant, K_d define in Equation 2b is the differentiators gain. This differentiator circuit only has current flowing in the input when there is change in V_i(t). When there is no change in the input voltage, no current will flow and the output voltage V_o(t) will be zero. The ideal differentiator circuit only produces an output when ever there is a change in the input signal. This is useful in control circuits where rapid response to a change in the control variable is necessary.

Another way of examining the circuit is to check its output gain response to sine waves of different frequencies. When the gain of these tests is represented in db and the frequency is plotted on a logarithmic scale, a Bode plot is produced. Bode plots are used to determine the stability of control systems and the frequency response of filter circuits.

To find the Bode plot of the ideal differentiator circuit, the first step is to take the Laplace transform of the input-output relationship of Equation 2a. In the Laplace domain, differentiation in time converts to multiplication by the complex variable s. (s represents the complex frequency - transient and sine response of a system.)

Taking the Laplace transform of 2a gives:

$$\begin{aligned} L(v_o(t)) = V_o(s) \quad L(v_i(t)) = V_i(s) \quad (a) \\ V_o(s) = -\ RC\,s\ V_i(s) \quad (b) \\ \frac{V_o(s)}{V_i(s)} = -\ RCs \quad (c) \end{aligned} \tag{3}$$

Equation 3c is the transfer function of the ideal differentiator circuit of Figure 1. To convert this to a Bode plot, we must replace the complex variable s with its imaginary part to find the change of the circuit's gain as frequency changes, the magnitude and phase shift of the transfer function can be found. The magnitude and phase of any complex quantity can be found from the following relationships:

$$|z| = \sqrt{Re(z)^2 + Im(z)^2} \qquad \phi = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right) \tag{4}$$

Where z = a complex value
Re(z) = the real part of z
Im(z) = the imaginary part of z
φ = the phase angle of z

The equations below show this theory applied to the ideal differentiator circuit.

$$\begin{aligned} A_v(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = -\ RCj\omega \\ |A_v(\omega)| = RC\omega \\ \phi = 180 + 90 = 270 = -90° \end{aligned} \tag{5}$$

The equations in (5) show that the gain of this circuit increases as the frequency increases. In fact, the circuit has an infinite gain to high frequency signals. The phase shift is a constant -90 degrees. This includes the 180 degree shift due to the inverting OP AMP configuration. To construct the Bode plot the gain must be converted to db by using the formula

$$db(\omega) = 20 \log[A_v(\omega)]$$

The plots below show the gain response of the ideal differentiator circuit. The phase shift is a constant -90 degree over the entire range of frequency. Notice that the gain of the ideal differentiator increases at a constant rate over the range of the plot. The gain goes up 20 db for every decade (power of 10) in frequency increase. The value of 20 db is x10 that of the initial gain value.

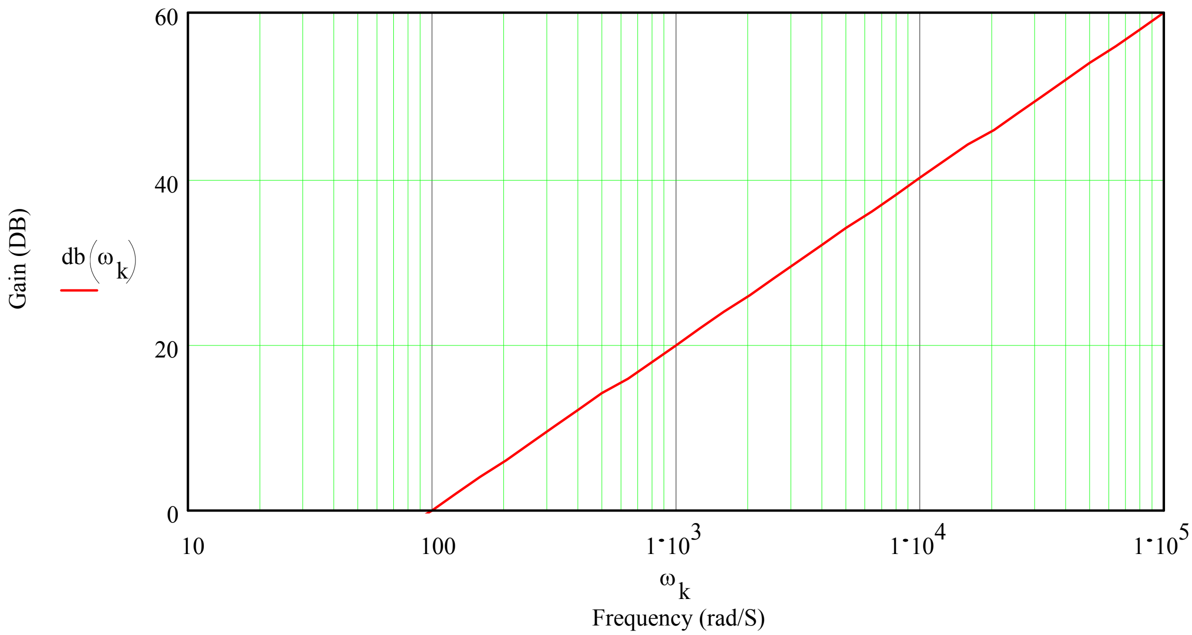

Figure 2. Ideal Differentiator Frequency Response.

The ideal differentiator is not a practical circuit. The infinite gain to high frequencies makes it impossible to construct because most noise signals are at high frequencies. Using the configuration shown is Figure 1 will cause the OP AMP circuit to go to saturation due to the high gain amplification of this electrical noise.

The bias current flowing in R_f also produces offset voltage error in the output. This voltage error can be minimized by adding an appropriately sized resistor in the non-inverting input of the OP AMP. The bias currents flowing through these resistors will develop a common mode voltage (same magnitude and phase) at the inputs to the OP AMP. The common mode voltage will not be amplified.

Note that the gain of the circuit reaches 0 db (1) at the frequency given by the value

$$\omega_c = 1/RC$$

Where ω_c = the cutoff frequency of the device in rad/s

Practical OP AMP Differentiators

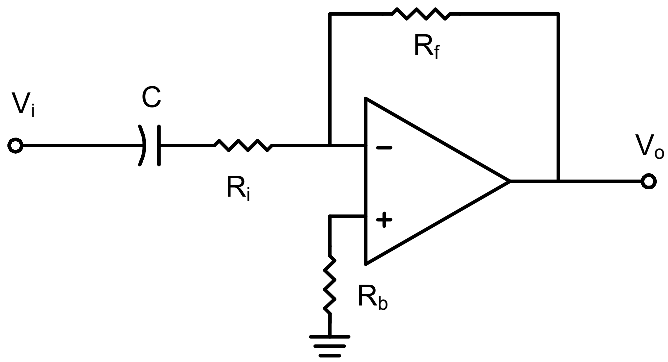

Figure 3. Practical Differentiator Circuit**.**

Figure 3 shows a practical integrator circuit that overcomes the limitations of the ideal circuit and still simulates the integrator action that is useful in control applications. This circuit is also known as an active highpass filter.

The value of resistor R_b is given by the parallel combination of the input and feed back resistances. In equation form this is:

$$R_b = R_i \;||\; R_f = R_f(R_i)/(R_f+R_i)$$

If the transfer characteristic of an inverting OP AMP circuit is written as the ratio of two impedances that have been converted using the rules of the Laplace transform, then the elements in the input branch can be combined using the rules of series impedances. The resulting value can then be substituted into the inverting gain formula and the transfer function written without a large amount of computation.
The following equations sketch out the mathematics used to find the transfer function for the practical integrator circuit.

$$\begin{aligned} v_o(t) &= \frac{1}{C}\int i_c(t)\,dt \quad (a) \\ V_o(s) &= \frac{1}{Cs}\; I_c(s) \quad (b) \\ Z_i(s) &= \frac{1}{Cs} + R_i \quad (c) \end{aligned} \tag{6}$$

Taking the Laplace of 6a gives 6b. The term 1/Cs can be interpreted as impedance and used in the series resistance formula to give the simplified value of the resistor and capacitor in the input branch $Z_i(s)$ in 6c.

Substituting the value of $Z_i(s)$ into the gain gives the transfer function as a function of the complex variable s.

$$\frac{V_o(s)}{V_i(s)} = \frac{-Z_f(s)}{Z_i(s)} \qquad Z_f(s) = R_f$$

$$\frac{V_o(s)}{V_i(s)} = \frac{-R_f}{R_i + \frac{1}{C\,s}} = \frac{-R_f\,C\,s}{R_i\,C\,s + 1} = A_v(s) \tag{7}$$

In the transfer function $A_v(s)$, the ratio of R_f/R_i, which is the same as the dc gain of an inverting OP AMP configuration, defines the gain the integrator has to a constant dc signal. This stabilizes the differentiator gain and eliminates the saturation effects of amplifier high frequency noise. The remainder of the transfer function defines the differentiator action of the circuit.

If the variable s is replaced by $j\omega$, and the magnitude and phase angle determined from procedures similar to the ideal case, the gain and phase shift can be found for any sinusoidal input frequency. These relationships are

$$|A_v(\omega)| = \left(\frac{R_f\,C\,\omega}{\sqrt{1 + R_i^2\,C^2\,\omega^2}}\right) \tag{8}$$

$$\phi(\omega) = 90° - \tan^{-1}(R_i\,C\,\omega)$$

The frequency in these equations is given in radians/sec. To convert the values to Hertz use the following relationship.

$$2\pi f = \omega$$

This function models the response of a high pass active filter. The ratio of R_f/R_i sets the gain in the pass band. The point where the gain begins to decrease is called the cutoff frequency. This is defined as the point where the gain is down 3 db from the gain in the pass band. A 3 db reduction in gain corresponds to a 0.707 reduction in the output voltage from the level in the pass band. This point is defined by

$f_c = 1/2\pi R_i C$ Hz

$\omega_c = 1/R_i C$ rad/S

After the cutoff point is reached, the gain of the circuit falls at a rate of 20 db/decade, just as in the ideal differentiator circuit. To use the practical differential as an differentiator, the highest frequency expected to be encountered in a control system must fall into this part of the circuit response. As "a rule of thumb" for designing a practical differentiator in a control system, set f_c to be 10 times the highest frequency encountered. The following figures show the Bode plots of gain and phase response for the practical differentiator circuit.

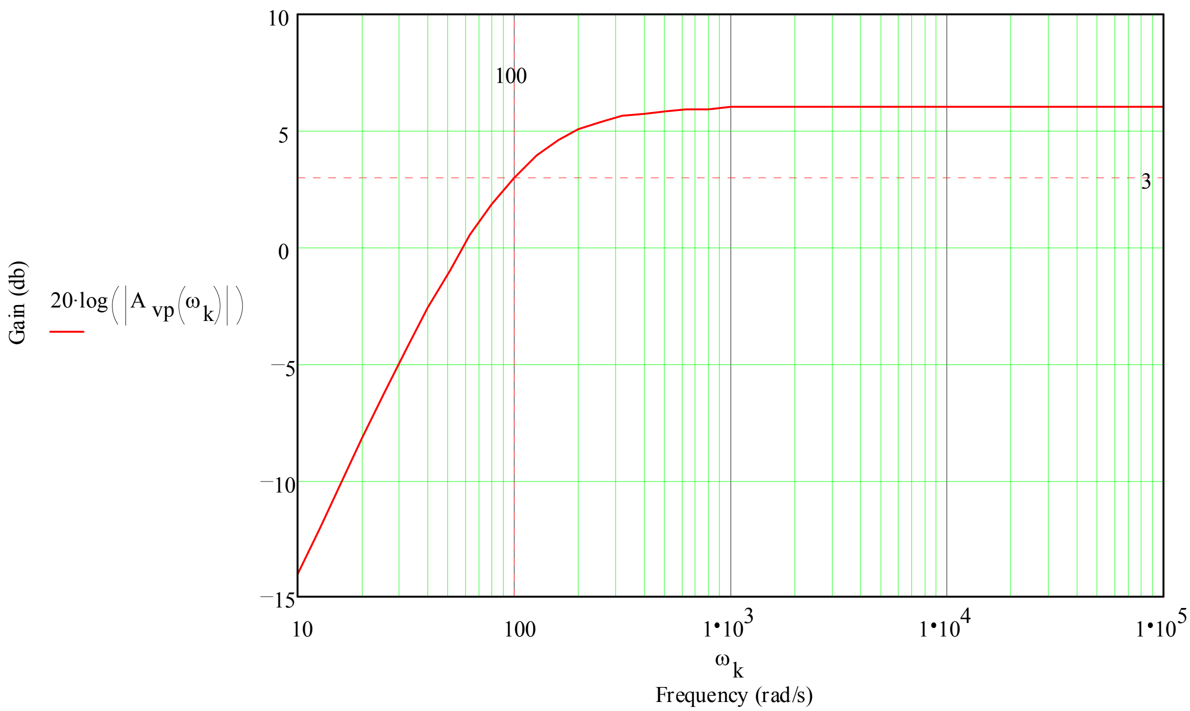

Figure 4. Gain Plot of Practical Differentiator Circuit Showing Cutoff Frequency.

In this plot the dc gain of the integrator is 2 (6 db) and the cutoff frequency is 100 rad/s (159.15 Hz). At the cutoff frequency, the output is down 3 db +3 db because the gain in the pass band is greater than 1. The phase plot includes the 180 degree phase shift due to the inverting action of the integrator circuit. The phase and the gain will both be important when the stability of control systems are examined.

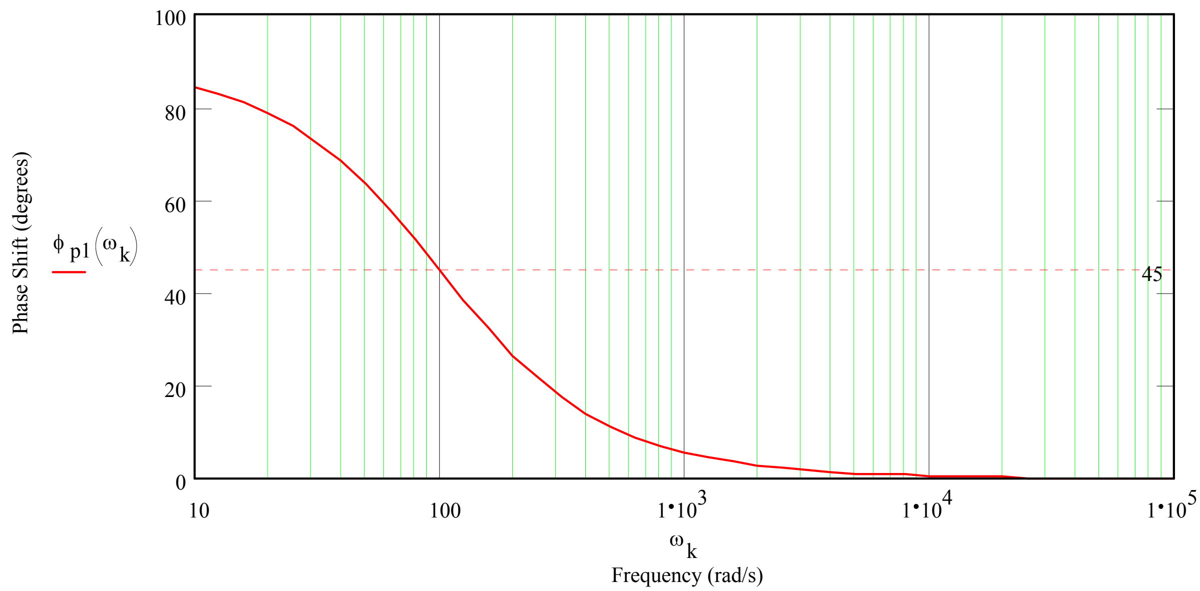

Figure 5. Phase Plot of Practical Differentiator Circuit.

In the pass band, well above the frequency of 100 rad/s, the only phase shift is from the inverting action of the OP AMP circuit. As the frequency decreases, the phase shift increases to a value of 45 degrees at the cutoff frequency. The phase shift continues to increase as the frequency decreases and will asymptotically approach 90 degrees for very low frequencies.

Time Response of a Practical Differentiator to Common Waveforms

An differentiator circuit simulates the mathematical operation of differentiation. Table 1 shows the results of applying common waveforms to the differentiator and how these wave forms can be modeled using mathematical formulae.

Table 1. Differentiator Time Response

Mathematical Model			Differentiator Circuit	
Function	Equation	Derivative (t)	Waveform In	Waveform Out
Constant	K	0	Square wave	0
line	Kt	K	Triangle	Square wave
sinusoidal	$A_{max}\sin(\omega t)$	$A_{max}\omega\cos(\omega t)$	Sine Wave	Shifted Sine

When an inverting differentiator is used, the sign of the output will be opposite of the mathematical model. If a positively increasing constant is applied to the practical integrator, (the positive half cycle of a triangle) the resulting output will be a negative part of a square wave.

To achieve the derivative action on these wave forms the input frequency must follow the x10 f_c rule introduced above. The gain will be increasing as the frequency increases as predicted by the Bode plots, so the magnitude of the output will increase until the pass band is reached. At this point the derivative action will stop and the signal will be amplified by a constant gain.

Design Project - Practical Integrator Circuits and Responses.

Design a practical differentiator circuit that has a dc gain in the pass band of 10 db and a cutoff frequency of 1000 Hz. Document all the design values for the lab report. Test the design and compare it to the expected theoretical values.

1.) To check the frequency response, apply a 1 $V_{p\text{-}p}$ sinusoidal ac signal to the input. Generate the test points for the Bode plot by applying the following signal frequencies:

100 Hz
200 Hz
500 Hz
700 Hz
1000 Hz
2.5 kHz
5 kHz
7 kHz
10 kHz
15 kHz
20 kHz

Maintain the input voltage constant and record the output voltage and phase shift for each of the listed frequencies. Use the input wave form as the reference for the phase measurements. Compute the differentiator gain using the formula:

$$db = 20 \log[V_o / V_i]$$

Using the formula for the practical differentiator derived above, compute the theoretical values of gain. Plot both the measured and theoretical values on the same semi-log (log scale on x-axis) plot. Discuss any deviations from the theoretical curve in the lab report.

Using the formula for the practical differentiator phase shift, compute the theoretical values of phase shift (in degrees) and plot both the measured and theoretical values on the same semilog plot (log scale on x-axis). Discuss any deviations from the theoretical curve in the lab report.

2a.) Apply a square wave signal with an amplitude of 1 Vp-p to the differentiator. Use the following frequencies: 200 Hz, 1000 Hz, and 25 kHz. Sketch the changes in the output waveform as the frequency changes and note any changes in amplitude and shape. Discuss these data in the lab report. Determine the frequency where the differentiator action stops.

2b.) Apply a triangle wave signal with amplitude of 1 Vp-p to the differentiator. Use the following frequencies: 200 Hz, 1000 Hz, and 25 kHz. Sketch the changes in the output waveform as the frequency changes and note any changes in amplitude and shape. Discuss these data in the lab report. Determine the frequency where the differentiator action stops.

3.) Derive the transfer function for the differentiator designed in the lab by substituting the design values into the final formula in (7). Include the simplified transfer function in the lab report along with a schematic that shows all the computed design values.